Figure Q35.10

areas of the Earth see a total eclipse, other areas see a partial eclipse, and most areas see no eclipse.

17. The display windows of some department stores are slanted slightly inward at the bottom. This is to decrease the glare from streetlights or the Sun, which would make it difficult for shoppers to see the display inside. Sketch a light ray reflecting off such a window to show how this technique works.

18. When two colors of light X and Y are sent through a glass prism, X is bent more than Y. Which color travels more slowly in the prism?

19. Why does the arc of a rainbow appear with red on top and violet on the bottom?

20. Under what conditions is a mirage formed? On a hot day, what are we seeing when we observe "water on the road"?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 35.1 The Nature of Light

Section 35.2 Measurements of the Speed of Light

1. The Apollo 11 astronauts set up a highly reflecting panel on the Moon's surface. The speed of light can be found by measuring the time it takes a laser beam to travel from Earth, reflect from the retroreflector, and return to Earth. If this interval is measured to be 2.51 s, what is the measured speed of light? Take the center-to-center distance from the Earth to the Moon to be 3.84×10^8 m, and do not neglect the sizes of the Earth and the Moon.

2. As a result of his observations, Roemer concluded that eclipses of Io by Jupiter were delayed by 22 min during a six-month period as the Earth moved from the point in its orbit where it is closest to Jupiter to the diametrically opposite point where it is farthest from Jupiter. Using 1.50×10^8 km as the average radius of the Earth's orbit around the Sun, calculate the speed of light from these data.

3. In an experiment to measure the speed of light using the apparatus of Fizeau (see Fig. 35.2), the distance between light source and mirror was 11.45 km and the wheel had 720 notches. The experimentally determined value of c was 2.998×10^8 m/s. Calculate the minimum angular speed of the wheel for this experiment.

4. Figure P35.4 shows an apparatus used to measure the speed distribution of gas molecules. It consists of two slotted rotating disks separated by a distance d, with the slots displaced by the angle θ. Suppose that the speed of light is measured by sending a light beam from the left through this apparatus. (a) Show that a light beam will be seen in the detector (that is, will make it through both slots) only if its speed is given by $c = \omega d/\theta$, where ω is the angular speed of the disks and θ is measured in radians. (b) What is the measured speed of light if the distance between the two slotted rotating disks is 2.50 m, the slot in the second disk is displaced 1/60 of 1° from the slot in the first disk, and the disks are rotating at 5 555 rev/s?

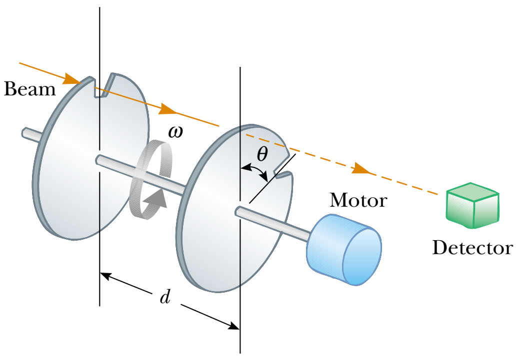


Figure P35.4

Section 35.3 The Ray Approximation in Geometric Optics
Section 35.4 Reflection
Section 35.5 Refraction
Section 35.6 Huygens's Principle

Note: In this section, if an index of refraction value is not given, refer to Table 35.1.

5. A narrow beam of sodium yellow light, with wavelength 589 nm in vacuum, is incident from air onto a smooth water surface at an angle $\theta_1 = 35.0°$. Determine the angle of refraction θ_2 and the wavelength of the light in water.

6. The wavelength of red helium–neon laser light in air is 632.8 nm. (a) What is its frequency? (b) What is its wavelength in glass that has an index of refraction of 1.50? (c) What is its speed in the glass?

7. An underwater scuba diver sees the Sun at an apparent angle of 45.0° from the vertical. What is the actual direction of the Sun?

8. A laser beam is incident at an angle of 30.0° from the vertical onto a solution of corn syrup in water. If the beam is refracted to 19.24° from the vertical, (a) what is the index of refraction of the syrup solution? Suppose that the light is red, with a vacuum wavelength of 632.8 nm. Find its (b) wavelength, (c) frequency, and (d) speed in the solution.

9. Find the speed of light in (a) flint glass, (b) water, and (c) cubic zirconia.

10. A light ray initially in water enters a transparent substance at an angle of incidence of 37.0°, and the transmitted ray is refracted at an angle of 25.0°. Calculate the speed of light in the transparent substance.

WEB **11.** A ray of light strikes a flat block of glass ($n = 1.50$) of thickness 2.00 cm at an angle of 30.0° with the normal. Trace the light beam through the glass, and find the angles of incidence and refraction at each surface.

12. Light of wavelength 436 nm in air enters a fishbowl filled with water and then exits through the crown glass wall of the container. What is the wavelength of the light (a) in the water and (b) in the glass?

13. An opaque cylindrical tank with an open top has a diameter of 3.00 m and is completely filled with water. When the setting Sun reaches an angle of 28.0° above the horizon, sunlight ceases to illuminate any part of the bottom of the tank. How deep is the tank?

14. The angle between the two mirrors illustrated in Figure P35.14 is a right angle. The beam of light in the vertical plane P strikes mirror 1 as shown. (a) Determine the distance that the reflected light beam travels before striking mirror 2. (b) In what direction does the light beam travel after being reflected from mirror 2?

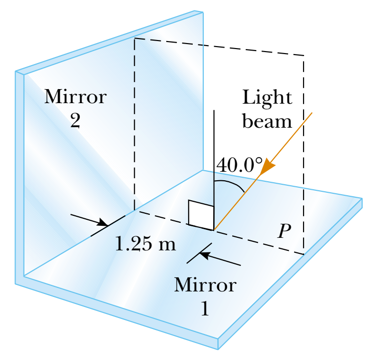


Figure P35.14

15. How many times will the incident beam shown in Figure P35.15 be reflected by each of the parallel mirrors?

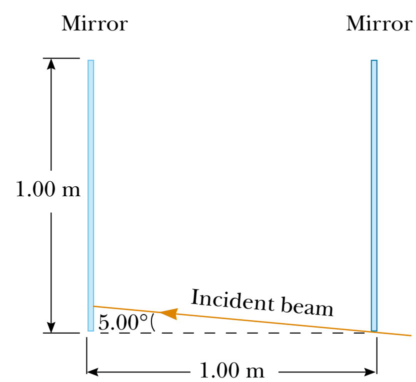


Figure P35.15

16. When the light illustrated in Figure P35.16 passes through the glass block, it is shifted laterally by the distance d. If $n = 1.50$, what is the value of d?

17. Find the time required for the light to pass through the glass block described in Problem 16.

18. The light beam shown in Figure P35.18 makes an angle of 20.0° with the normal line NN' in the linseed oil. Determine the angles θ and θ'. (The index of refraction for linseed oil is 1.48.)

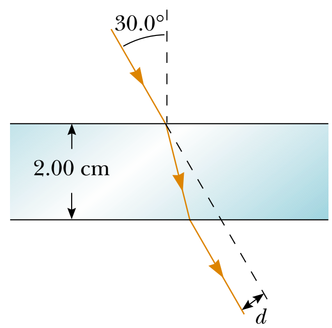


Figure P35.16 Problems 16 and 17.

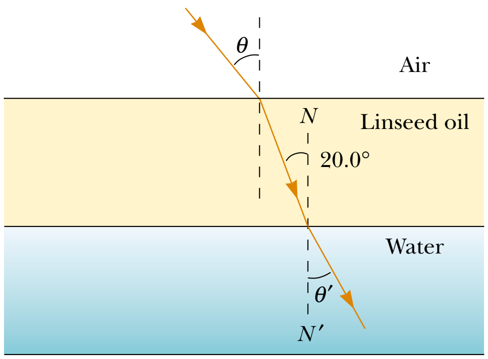


Figure P35.18

19. Two light pulses are emitted simultaneously from a source. Both pulses travel to a detector, but one first passes through 6.20 m of ice. Determine the difference in the pulses' times of arrival at the detector.

20. When you look through a window, by how much time is the light you see delayed by having to go through glass instead of air? Make an order-of-magnitude estimate on the basis of data you specify. By how many wavelengths is it delayed?

21. Light passes from air into flint glass. (a) What angle of incidence must the light have if the component of its velocity perpendicular to the interface is to remain constant? (b) Can the component of velocity parallel to the interface remain constant during refraction?

22. The reflecting surfaces of two intersecting flat mirrors are at an angle of θ ($0° < \theta < 90°$), as shown in Figure P35.22. If a light ray strikes the horizontal mirror, show that the emerging ray will intersect the incident ray at an angle of $\beta = 180° - 2\theta$.

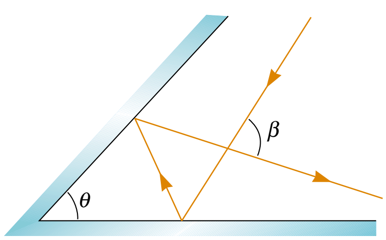


Figure P35.22

23. A light ray enters the atmosphere of a planet and descends vertically 20.0 km to the surface. The index of refraction where the light enters the atmosphere is 1.000, and it increases linearly to the surface where it has a value of 1.005. (a) How long does it take the ray to traverse this path? (b) Compare this to the time it takes in the absence of an atmosphere.

24. A light ray enters the atmosphere of a planet and descends vertically to the surface a distance h. The index of refraction where the light enters the atmosphere is 1.000, and it increases linearly to the surface where it has a value of n. (a) How long does it take the ray to traverse this path? (b) Compare this to the time it takes in the absence of an atmosphere.

Section 35.7 Dispersion and Prisms

25. A narrow white light beam is incident on a block of fused quartz at an angle of 30.0°. Find the angular width of the light beam inside the quartz.

26. A ray of light strikes the midpoint of one face of an equiangular glass prism ($n = 1.50$) at an angle of incidence of 30.0°. Trace the path of the light ray through the glass, and find the angles of incidence and refraction at each surface.

27. A prism that has an apex angle of 50.0° is made of cubic zirconia, with $n = 2.20$. What is its angle of minimum deviation?

28. Light with a wavelength of 700 nm is incident on the face of a fused quartz prism at an angle of 75.0° (with respect to the normal to the surface). The apex angle of the prism is 60.0°. Using the value of n from Figure 35.20, calculate the angle (a) of refraction at this first surface, (b) of incidence at the second surface, (c) of refraction at the second surface, and (d) between the incident and emerging rays.

WEB **29.** The index of refraction for violet light in silica flint glass is 1.66, and that for red light is 1.62. What is the angular dispersion of visible light passing through a prism of apex angle 60.0° if the angle of incidence is 50.0°? (See Fig. P35.29.)

30. Show that if the apex angle Φ of a prism is small, an approximate value for the angle of minimum deviation is $\delta_{min} = (n - 1)\Phi$.

31. A triangular glass prism with an apex angle of $\Phi = 60.0°$ has an index of refraction $n = 1.50$ (Fig. P35.31). What is the smallest angle of incidence θ_1 for which a light ray can emerge from the other side?

32. A triangular glass prism with an apex angle of Φ has an index of refraction n (Fig. P35.31). What is the smallest angle of incidence θ_1 for which a light ray can emerge from the other side?

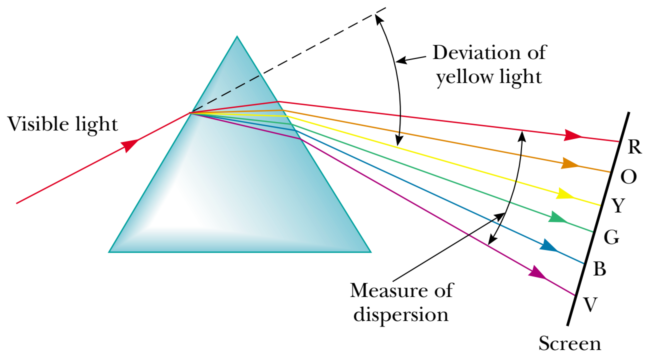


Figure P35.29

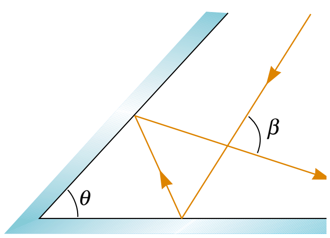


Figure P35.31

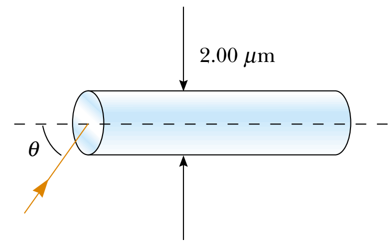


Figure P35.38

33. An experimental apparatus includes a prism made of sodium chloride. The angle of minimum deviation for light of wavelength 589 nm is to be 10.0°. What is the required apex angle of the prism?

34. A triangular glass prism with an apex angle of 60.0° has an index of refraction of 1.50. (a) Show that if its angle of incidence on the first surface is $\theta_1 = 48.6°$, light will pass symmetrically through the prism, as shown in Figure 35.26. (b) Find the angle of deviation δ_{min} for $\theta_1 = 48.6°$. (c) Find the angle of deviation if the angle of incidence on the first surface is 45.6°. (d) Find the angle of deviation if $\theta_1 = 51.6°$.

Section 35.8 Total Internal Reflection

35. For 589-nm light, calculate the critical angle for the following materials surrounded by air: (a) diamond, (b) flint glass, and (c) ice.

36. Repeat Problem 35 for the situation in which the materials are surrounded by water.

37. Consider a common mirage formed by super-heated air just above a roadway. A truck driver whose eyes are 2.00 m above the road, where $n = 1.000\,3$, looks forward. She perceives the illusion of a patch of water ahead on the road, where her line of sight makes an angle of 1.20° below the horizontal. Find the index of refraction of the air just above the road surface. (*Hint:* Treat this as a problem in total internal reflection.)

38. Determine the maximum angle θ for which the light rays incident on the end of the pipe shown in Figure P35.38 are subject to total internal reflection along the walls of the pipe. Assume that the pipe has an index of refraction of 1.36 and the outside medium is air.

39. A glass fiber ($n = 1.50$) is submerged in water ($n = 1.33$). What is the critical angle for light to stay inside the optical fiber?

40. A glass cube is placed on a newspaper, which rests on a table. A person reads all of the words the cube covers, through all of one vertical side. Determine the maximum possible index of refraction of the glass.

41. A large Lucite cube ($n = 1.59$) has a small air bubble (a defect in the casting process) below one surface. When a penny (diameter, 1.90 cm) is placed directly over the bubble on the outside of the cube, one cannot see the bubble by looking down into the cube at any angle. However, when a dime (diameter, 1.75 cm) is placed directly over it, one can see the bubble by looking down into the cube. What is the range of the possible depths of the air bubble beneath the surface?

42. A room contains air in which the speed of sound is 343 m/s. The walls of the room are made of concrete, in which the speed of sound is 1 850 m/s. (a) Find the critical angle for total internal reflection of sound at the concrete–air boundary. (b) In which medium must the sound be traveling to undergo total internal reflection? (c) "A bare concrete wall is a highly efficient mirror for sound." Give evidence for or against this statement.

43. In about 1965, engineers at the Toro Company invented a gasoline gauge for small engines, diagrammed in Figure P35.43. The gauge has no moving parts. It consists of a flat slab of transparent plastic fitting vertically into a slot in the cap on the gas tank. None of the plastic has a reflective coating. The plastic projects from the horizontal top down nearly to the bottom of the opaque tank. Its lower edge is cut with facets making angles of 45° with the horizontal. A lawnmower operator looks down from above and sees a boundary between bright and dark on the gauge. The location of the boundary, across the width of the plastic, indicates the quantity of gasoline in the tank. Explain how the gauge works. Explain the design requirements, if any, for the index of refraction of the plastic.

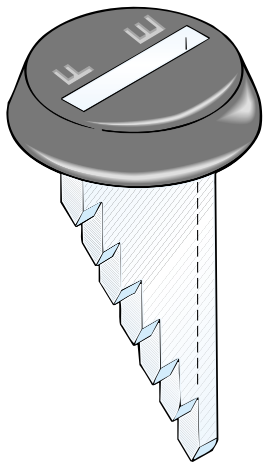


Figure P35.43

(Optional)
Section 35.9 Fermat's Principle

44. The shoreline of a lake runs from east to west. A swimmer gets into trouble 20.0 m out from shore and 26.0 m to the east of a lifeguard, whose station is 16.0 m in from the shoreline. The lifeguard takes a negligible amount of time to accelerate. He can run at 7.00 m/s and swim at 1.40 m/s. To reach the swimmer as quickly as possible, in what direction should the lifeguard start running? You will need to solve a transcendental equation numerically.

ADDITIONAL PROBLEMS

45. A narrow beam of light is incident from air onto a glass surface with an index of refraction of 1.56. Find the angle of incidence for which the corresponding angle of refraction is one half the angle of incidence. (*Hint:* You might want to use the trigonometric identity $\sin 2\theta = 2 \sin \theta \cos \theta$.)

46. (a) Consider a horizontal interface between air above and glass with an index of 1.55 below. Draw a light ray incident from the air at an angle of incidence of 30.0°. Determine the angles of the reflected and refracted rays and show them on the diagram. (b) Suppose instead that the light ray is incident from the glass at an angle of incidence of 30.0°. Determine the angles of the reflected and refracted rays and show all three rays on a new diagram. (c) For rays incident from the air onto the air–glass surface, determine and tabulate the angles of reflection and refraction for all the angles of incidence at 10.0° intervals from 0 to 90.0°. (d) Do the same for light rays traveling up to the interface through the glass.

WEB 47. A small underwater pool light is 1.00 m below the surface. The light emerging from the water forms a circle on the water's surface. What is the diameter of this circle?

48. One technique for measuring the angle of a prism is shown in Figure P35.48. A parallel beam of light is directed on the angle so that the beam reflects from opposite sides. Show that the angular separation of the two beams is given by $B = 2A$.

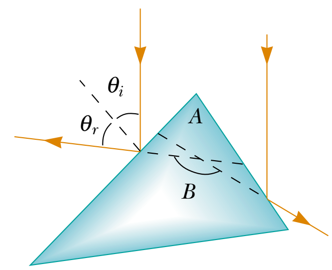


Figure P35.48

49. The walls of a prison cell are perpendicular to the four cardinal compass directions. On the first day of spring, light from the rising Sun enters a rectangular window in the eastern wall. The light traverses 2.37 m horizontally to shine perpendicularly on the wall opposite the window. A young prisoner observes the patch of light moving across this western wall and for the first time forms his own understanding of the rotation of the Earth. (a) With what speed does the illuminated rectangle move? (b) The prisoner holds a small square mirror flat against the wall at one corner of the rectangle of light. The mirror reflects light back to a spot on the eastern wall close beside the window. How fast does the smaller square of light move across that wall? (c) Seen from a latitude of 40.0° north, the rising Sun moves through the sky along a line making a 50.0° angle with the southeastern horizon. In what direction does the rectangular patch of light on the western wall of the prisoner's cell move? (d) In what direction does the smaller square of light on the eastern wall move?

50. The laws of refraction and reflection are the same for sound as for light. The speed of sound in air is 340 m/s, and that of sound in water is 1 510 m/s. If a sound wave approaches a plane water surface at an angle of incidence of 12.0°, what is the angle of refraction?

51. Cold sodium atoms (near absolute zero) in a state called a *Bose–Einstein condensate* can slow the speed of light from its normally high value to a speed approaching that of an automobile in a city. The speed of light in one such medium was recorded as 61.15 km/h. (a) Find the index of refraction of this medium. (b) What is the critical angle for total internal reflection if the condensate is surrounded by vacuum?

52. A narrow beam of white light is incident at 25.0° onto a slab of heavy flint glass 5.00 cm thick. The indices of

refraction of the glass at wavelengths of 400 nm and 700 nm are 1.689 and 1.642, respectively. Find the width of the visible beam as it emerges from the slab.

53. A hiker stands on a mountain peak near sunset and observes a rainbow caused by water droplets in the air 8.00 km away. The valley is 2.00 km below the mountain peak and entirely flat. What fraction of the complete circular arc of the rainbow is visible to the hiker? (See Fig. 35.25.)

54. A fish is at a depth d under water. Take the index of refraction of water as 4/3. Show that when the fish is viewed at an angle of refraction θ_1, the apparent depth z of the fish is

$$z = \frac{3d\cos\theta_1}{\sqrt{7 + 9\cos^2\theta_1}}$$

WEB **55.** A laser beam strikes one end of a slab of material, as shown in Figure P35.55. The index of refraction of the slab is 1.48. Determine the number of internal reflections of the beam before it emerges from the opposite end of the slab.

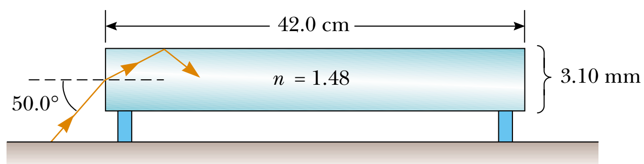


Figure P35.55

56. When light is normally incident on the interface between two transparent optical media, the intensity of the reflected light is given by the expression

$$S_1' = \left(\frac{n_2 - n_1}{n_2 + n_1}\right)^2 S_1$$

In this equation, S_1 represents the average magnitude of the Poynting vector in the incident light (the incident intensity), S_1' is the reflected intensity, and n_1 and n_2 are the refractive indices of the two media. (a) What fraction of the incident intensity is reflected for 589-nm light normally incident on an interface between air and crown glass? (b) In part (a), does it matter whether the light is in the air or in the glass as it strikes the interface? (c) A Bose–Einstein condensate (see Problem 51) has an index of refraction of 1.76×10^7. Find the percent reflection for light falling perpendicularly on its surface. What would the condensate look like?

57. Refer to Problem 56 for a description of the reflected intensity of light normally incident on an interface between two transparent media. (a) When light is normally incident on an interface between vacuum and a transparent medium of index n, show that the intensity S_2 of the transmitted light is given by the expression $S_2/S_1 = 4n/(n+1)^2$. (b) Light travels perpendicularly through a diamond slab, surrounded by air, with parallel surfaces of entry and exit. Apply the transmission fraction in part (a) to find the approximate overall transmission through the slab of diamond as a percentage. Ignore light reflected back and forth within the slab.

58. This problem builds upon the results of Problems 56 and 57. Light travels perpendicularly through a diamond slab, surrounded by air, with parallel surfaces of entry and exit. What fraction of the incident intensity is the intensity of the transmitted light? Include the effects of light reflected back and forth inside the slab.

59. The light beam shown in Figure P35.59 strikes surface 2 at the critical angle. Determine the angle of incidence, θ_1.

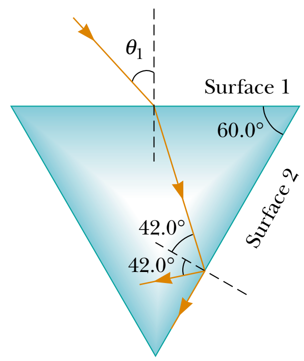


Figure P35.59

60. A 4.00-m-long pole stands vertically in a lake having a depth of 2.00 m. When the Sun is 40.0° above the horizontal, determine the length of the pole's shadow on the bottom of the lake. Take the index of refraction for water to be 1.33.

WEB **61.** A light ray of wavelength 589 nm is incident at an angle θ on the top surface of a block of polystyrene, as shown in Figure P35.61. (a) Find the maximum value of θ for which the refracted ray undergoes total internal reflec-

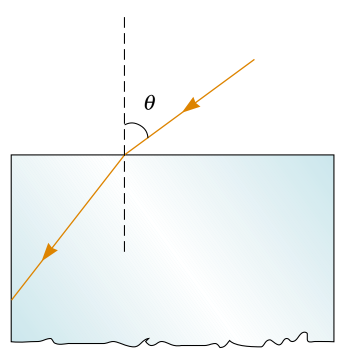


Figure P35.61

tion at the left vertical face of the block. Repeat the calculation for the case in which the polystyrene block is immersed in (b) water and (c) carbon disulfide.

62. A ray of light passes from air into water. For its deviation angle $\delta = |\theta_1 - \theta_2|$ to be 10.0°, what must be its angle of incidence?

63. A shallow glass dish is 4.00 cm wide at the bottom, as shown in Figure P35.63. When an observer's eye is positioned as shown, the observer sees the edge of the bottom of the empty dish. When this dish is filled with water, the observer sees the center of the bottom of the dish. Find the height of the dish.

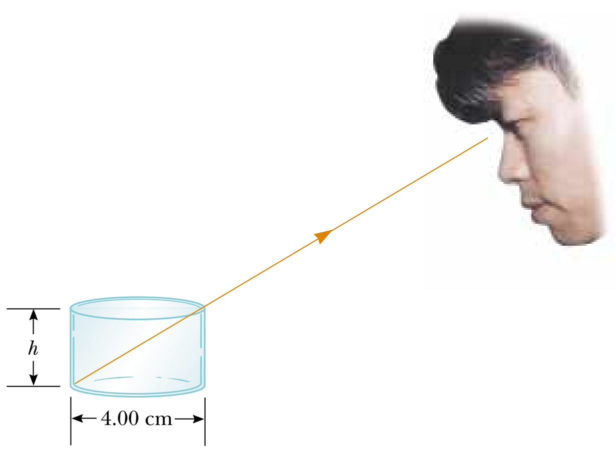


Figure P35.63

64. A material having an index of refraction n is surrounded by a vacuum and is in the shape of a quarter circle of radius R (Fig. P35.64). A light ray parallel to the base of the material is incident from the left at a distance of L above the base and emerges out of the material at the angle θ. Determine an expression for θ.

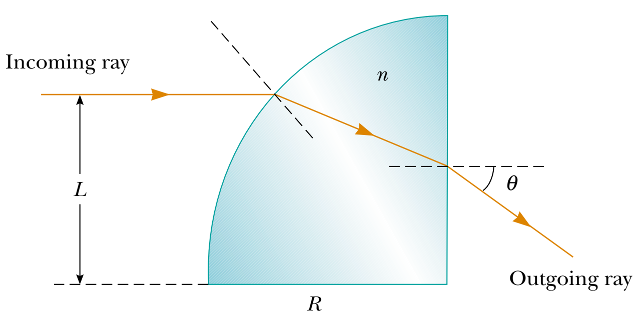


Figure P35.64

65. Derive the law of reflection (Eq. 35.2) from Fermat's principle of least time. (See the procedure outlined in Section 35.9 for the derivation of the law of refraction from Fermat's principle.)

66. A transparent cylinder of radius $R = 2.00$ m has a mirrored surface on its right half, as shown in Figure P35.66. A light ray traveling in air is incident on the left side of the cylinder. The incident light ray and exiting light ray are parallel and $d = 2.00$ m. Determine the index of refraction of the material.

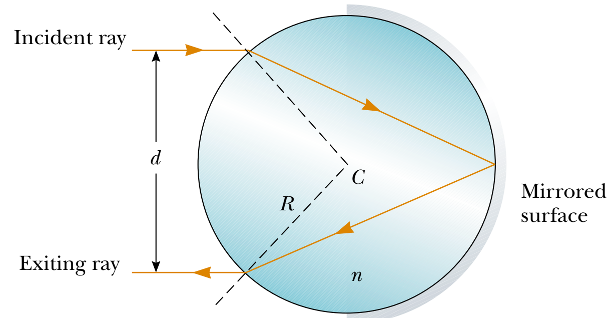


Figure P35.66

67. A. H. Pfund's method for measuring the index of refraction of glass is illustrated in Figure P35.67. One face of a slab of thickness t is painted white, and a small hole scraped clear at point P serves as a source of diverging rays when the slab is illuminated from below. Ray PBB' strikes the clear surface at the critical angle and is totally reflected, as are rays such as PCC'. Rays such as PAA' emerge from the clear surface. On the painted surface there appears a dark circle of diameter d, surrounded by an illuminated region, or halo. (a) Derive a formula for n in terms of the measured quantities d and t. (b) What is the diameter of the dark circle if $n = 1.52$ for a slab 0.600 cm thick? (c) If white light is used, the critical angle depends on color caused by dispersion. Is the inner edge of the white halo tinged with red light or violet light? Explain.

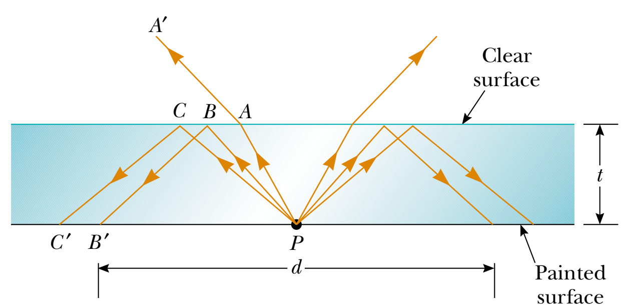


Figure P35.67

68. A light ray traveling in air is incident on one face of a right-angle prism with an index of refraction of $n = 1.50$, as shown in Figure P35.68, and the ray follows the path shown in the figure. If $\theta = 60.0°$ and the base of the prism is mirrored, what is the angle ϕ made by

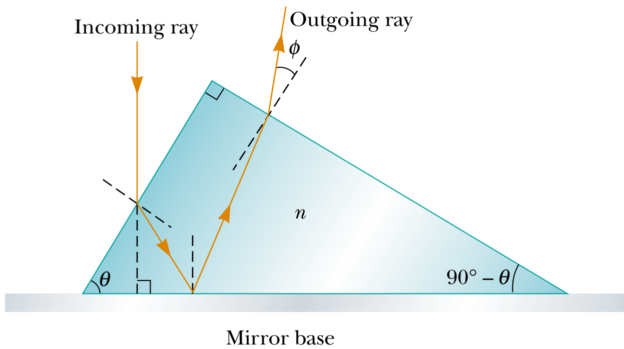


Figure P35.68

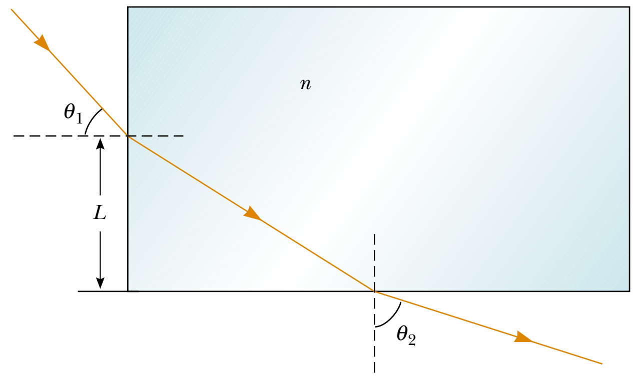


Figure P35.69

the outgoing ray with the normal to the right face of the prism?

69. A light ray enters a rectangular block of plastic at an angle of $\theta_1 = 45.0°$ and emerges at an angle of $\theta_2 = 76.0°$, as shown in Figure P35.69. (a) Determine the index of refraction for the plastic. (b) If the light ray enters the plastic at a point $L = 50.0$ cm from the bottom edge, how long does it take the light ray to travel through the plastic?

70. Students allow a narrow beam of laser light to strike a water surface. They arrange to measure the angle of refraction for selected angles of incidence and record the data shown in the accompanying table. Use the data to verify Snell's law of refraction by plotting the sine of the angle of incidence versus the sine of the angle of refraction. Use the resulting plot to deduce the index of refraction of water.

Angle of Incidence (degrees)	Angle of Refraction (degrees)
10.0	7.5
20.0	15.1
30.0	22.3
40.0	28.7
50.0	35.2
60.0	40.3
70.0	45.3
80.0	47.7

Answers to Quick Quizzes

35.1 Beams ② and ④ are reflected; beams ③ and ⑤ are refracted.

35.2 Fused quartz. An ideal lens would have an index of refraction that does not vary with wavelength so that all colors would be bent through the same angle by the lens. Of the three choices, fused quartz has the least variation in n across the visible spectrum. Thus, it is the best choice for a single-element lens.

35.3 The two rays on the right result from total internal reflection at the right face of the prism. Because all of the light in these rays is reflected (rather than partly refracted), these two rays are brightest. The light from the other three rays is divided into reflected and refracted parts.

Puzzler

Most car headlights have lines across their faces, like those shown here. Without these lines, the headlights either would not function properly or would be much more likely to break from the jarring of the car on a bumpy road. What is the purpose of the lines? *(George Semple)*

chapter

36

Geometric Optics

Chapter Outline

36.1 Images Formed by Flat Mirrors
36.2 Images Formed by Spherical Mirrors
36.3 Images Formed by Refraction
36.4 Thin Lenses
36.5 *(Optional)* Lens Aberrations
36.6 *(Optional)* The Camera
36.7 *(Optional)* The Eye
36.8 *(Optional)* The Simple Magnifier
36.9 *(Optional)* The Compound Microscope
36.10 *(Optional)* The Telescope

This chapter is concerned with the images that result when spherical waves fall on flat and spherical surfaces. We find that images can be formed either by reflection or by refraction and that mirrors and lenses work because of reflection and refraction. We continue to use the ray approximation and to assume that light travels in straight lines. Both of these steps lead to valid predictions in the field called *geometric optics.* In subsequent chapters, we shall concern ourselves with interference and diffraction effects—the objects of study in the field of *wave optics.*

36.1 IMAGES FORMED BY FLAT MIRRORS

14.6

We begin by considering the simplest possible mirror, the flat mirror. Consider a point source of light placed at O in Figure 36.1, a distance p in front of a flat mirror. The distance p is called the **object distance.** Light rays leave the source and are reflected from the mirror. Upon reflection, the rays continue to diverge (spread apart), but they appear to the viewer to come from a point I behind the mirror. Point I is called the **image** of the object at O. Regardless of the system under study, we always locate images by extending diverging rays back to a point from which they appear to diverge. **Images are located either at the point from which rays of light *actually* diverge or at the point from which they *appear* to diverge.** Because the rays in Figure 36.1 appear to originate at I, which is a distance q behind the mirror, this is the location of the image. The distance q is called the **image distance.**

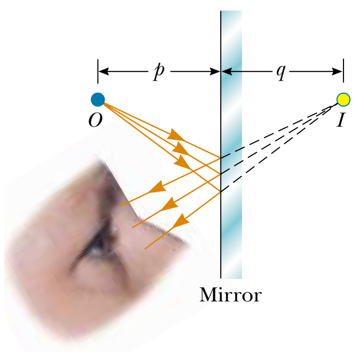


Figure 36.1 An image formed by reflection from a flat mirror. The image point I is located behind the mirror a perpendicular distance q from the mirror (the image distance). Study of Figure 36.2 shows that this image distance has the same magnitude as the object distance p.

Images are classified as real or virtual. **A real image is formed when light rays pass through and diverge from the image point; a virtual image is formed when the light rays do not pass through the image point but appear to diverge from that point.** The image formed by the mirror in Figure 36.1 is virtual. The image of an object seen in a flat mirror is always virtual. Real images can be displayed on a screen (as at a movie), but virtual images cannot be displayed on a screen.

We can use the simple geometric techniques shown in Figure 36.2 to examine the properties of the images formed by flat mirrors. Even though an infinite number of light rays leave each point on the object, we need to follow only two of them to determine where an image is formed. One of those rays starts at P, follows a horizontal path to the mirror, and reflects back on itself. The second ray follows the oblique path PR and reflects as shown, according to the law of reflection. An observer in front of the mirror would trace the two reflected rays back to the point at which they appear to have originated, which is point P' behind the mirror. A continuation of this process for points other than P on the object would result in a virtual image (represented by a yellow arrow) behind the mirror. Because triangles PQR and $P'QR$ are congruent, $PQ = P'Q$. We conclude that **the image formed by an object placed in front of a flat mirror is as far behind the mirror as the object is in front of the mirror.**

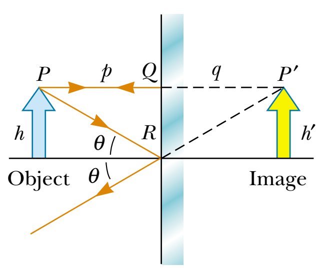


Figure 36.2 A geometric construction that is used to locate the image of an object placed in front of a flat mirror. Because the triangles PQR and $P'QR$ are congruent, $|p| = |q|$ and $h = h'$.

Geometry also reveals that the object height h equals the image height h'. Let us define **lateral magnification** M as follows:

Lateral magnification

$$M \equiv \frac{\text{Image height}}{\text{Object height}} = \frac{h'}{h} \tag{36.1}$$

Mt. Hood reflected in Trillium Lake. Why is the image inverted and the same size as the mountain?

QuickLab

View yourself in a full-length mirror. Standing close to the mirror, place one piece of tape at the top of the image of your head and another piece at the very bottom of the image of your feet. Now step back a few meters and observe your image. How big is it relative to its original size? How does the distance between the pieces of tape compare with your actual height? You may want to refer to Problem 3.

This is a general definition of the lateral magnification for any type of mirror. For a flat mirror, $M = 1$ because $h' = h$.

Finally, note that a flat mirror produces an image that has an *apparent* left–right reversal. You can see this reversal by standing in front of a mirror and raising your right hand, as shown in Figure 36.3. The image you see raises its left hand. Likewise, your hair appears to be parted on the side opposite your real part, and a mole on your right cheek appears to be on your left cheek.

This reversal is not *actually* a left–right reversal. Imagine, for example, lying on your left side on the floor, with your body parallel to the mirror surface. Now your head is on the left and your feet are on the right. If you shake your feet, the image does not shake its head! If you raise your right hand, however, the image again raises its left hand. Thus, the mirror again appears to produce a left–right reversal but in the up–down direction!

The reversal is actually a *front–back reversal,* caused by the light rays going forward toward the mirror and then reflecting back from it. An interesting exercise is to stand in front of a mirror while holding an overhead transparency in front of you so that you can read the writing on the transparency. You will be able to read the writing on the image of the transparency, also. You may have had a similar experience if you have attached a transparent decal with words on it to the rear window of your car. If the decal can be read from outside the car, you can also read it when looking into your rearview mirror from inside the car.

We conclude that the image that is formed by a flat mirror has the following properties.

Figure 36.3 The image in the mirror of a person's right hand is reversed front to back. This makes the right hand appear to be a left hand. Notice that the thumb is on the left side of both real hands and on the left side of the image. That the thumb is not on the right side of the image indicates that there is no left-to-right reversal.

- The image is as far behind the mirror as the object is in front of the mirror.
- The image is unmagnified, virtual, and upright. (By *upright* we mean that, if the object arrow points upward as in Figure 36.2, so does the image arrow.)
- The image has front–back reversal.

Quick Quiz 36.1

In the overhead view of Figure 36.4, the image of the stone seen by observer 1 is at C. Where does observer 2 see the image—at A, at B, at C, at D, at E, or not at all?

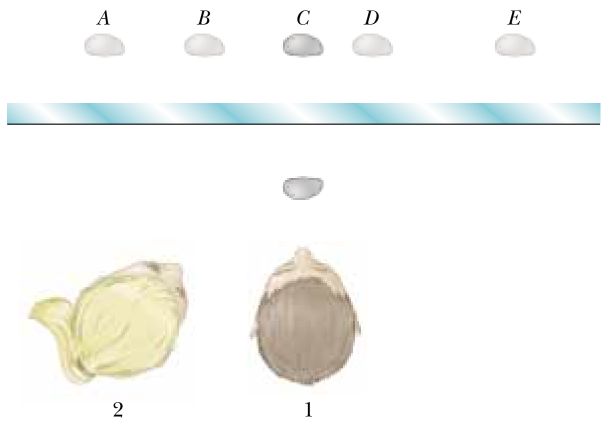


Figure 36.4

CONCEPTUAL EXAMPLE 36.1 Multiple Images Formed by Two Mirrors

Two flat mirrors are at right angles to each other, as illustrated in Figure 36.5, and an object is placed at point O. In this situation, multiple images are formed. Locate the positions of these images.

Solution The image of the object is at I_1 in mirror 1 and at I_2 in mirror 2. In addition, a third image is formed at I_3. This third image is the image of I_1 in mirror 2 or, equivalently, the image of I_2 in mirror 1. That is, the image at I_1 (or I_2) serves as the object for I_3. Note that to form this image at I_3, the rays reflect twice after leaving the object at O.

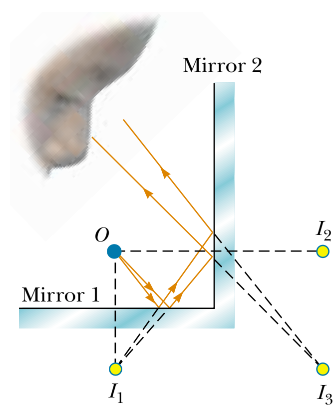


Figure 36.5 When an object is placed in front of two mutually perpendicular mirrors as shown, three images are formed.

CONCEPTUAL EXAMPLE 36.2 The Levitated Professor

The professor in the box shown in Figure 36.6 appears to be balancing himself on a few fingers, with his feet off the floor. He can maintain this position for a long time, and he appears to defy gravity. How was this illusion created?

Solution This is one of many magicians' optical illusions that make use of a mirror. The box in which the professor stands is a cubical frame that contains a flat vertical mirror positioned in a diagonal plane of the frame. The professor straddles the mirror so that one foot, which you see, is in front of the mirror, and one foot, which you cannot see, is behind the mirror. When he raises the foot in front of the mirror, the reflection of that foot also rises, so he appears to float in air.

Figure 36.6 An optical illusion.

CONCEPTUAL EXAMPLE 36.3 The Tilting Rearview Mirror

Most rearview mirrors in cars have a day setting and a night setting. The night setting greatly diminishes the intensity of the image in order that lights from trailing vehicles do not blind the driver. How does such a mirror work?

Solution Figure 36.7 shows a cross-sectional view of a rearview mirror for each setting. The unit consists of a reflective coating on the back of a wedge of glass. In the day setting (Fig. 36.7a), the light from an object behind the car strikes the glass wedge at point 1. Most of the light enters the wedge, refracting as it crosses the front surface, and reflects from the back surface to return to the front surface, where it is refracted again as it re-enters the air as ray B (for *bright*). In addition, a small portion of the light is reflected at the front surface of the glass, as indicated by ray D (for *dim*).

This dim reflected light is responsible for the image that is observed when the mirror is in the night setting (Fig. 36.7b). In this case, the wedge is rotated so that the path followed by the bright light (ray B) does not lead to the eye. Instead, the dim light reflected from the front surface of the wedge travels to the eye, and the brightness of trailing headlights does not become a hazard.

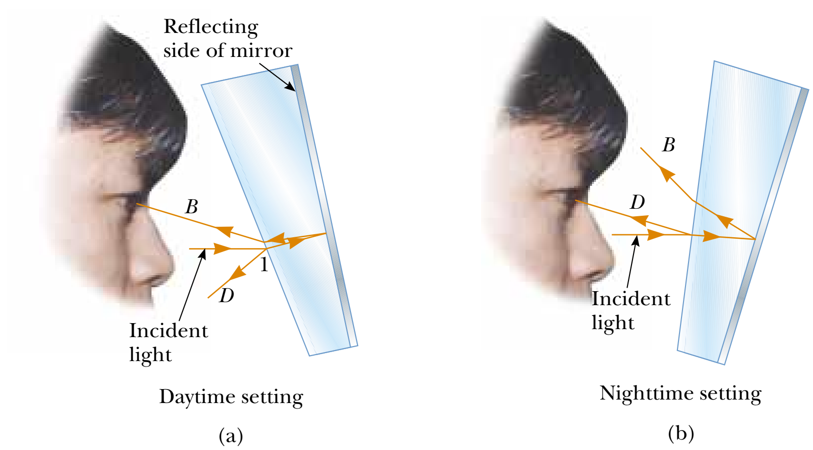


Figure 36.7 Cross-sectional views of a rearview mirror. (a) With the day setting, the silvered back surface of the mirror reflects a bright ray B into the driver's eyes. (b) With the night setting, the glass of the unsilvered front surface of the mirror reflects a dim ray D into the driver's eyes.

36.2 IMAGES FORMED BY SPHERICAL MIRRORS

Concave Mirrors

14.7 A **spherical mirror,** as its name implies, has the shape of a section of a sphere. This type of mirror focuses incoming parallel rays to a point, as demonstrated by the colored light rays in Figure 36.8. Figure 36.9a shows a cross-section of a spherical mirror, with its surface represented by the solid, curved black line. (The blue band represents the structural support for the mirrored surface, such as a curved piece of glass on which the silvered surface is deposited.) Such a mirror, in which light is reflected from the inner, concave surface, is called a **concave mirror.** The mirror has a radius of curvature R, and its center of curvature is point C. Point V is the center of the spherical section, and a line through C and V is called the **principal axis** of the mirror.

Now consider a point source of light placed at point O in Figure 36.9b, where O is any point on the principal axis to the left of C. Two diverging rays that originate at O are shown. After reflecting from the mirror, these rays converge (come together) at the image point I. They then continue to diverge from I as if an object were there. As a result, we have at point I a real image of the light source at O.

We shall consider in this section only rays that diverge from the object and make a small angle with the principal axis. Such rays are called **paraxial rays.** All

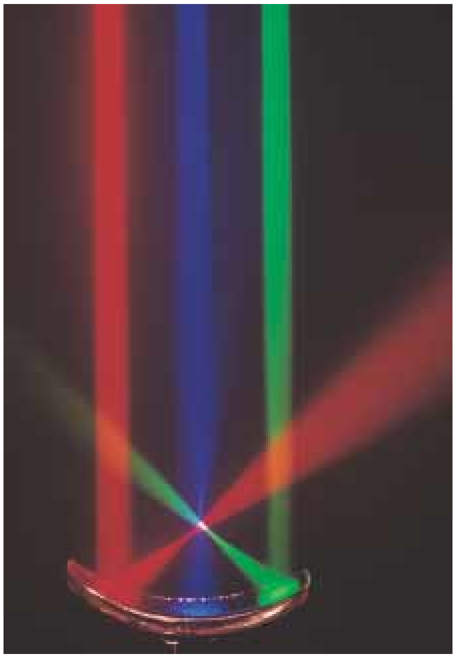


Figure 36.8 Red, blue, and green light rays are reflected by a curved mirror. Note that the point where the three colors meet is white.

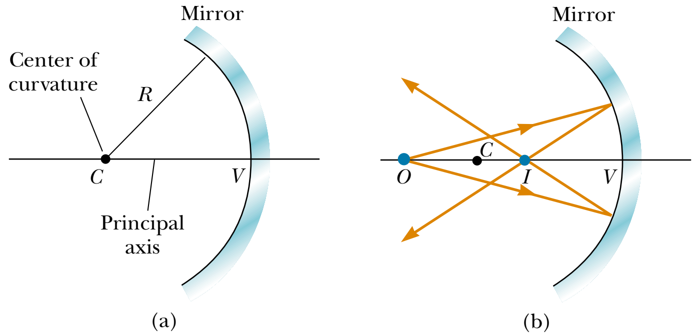


Figure 36.9 (a) A concave mirror of radius R. The center of curvature C is located on the principal axis. (b) A point object placed at O in front of a concave spherical mirror of radius R, where O is any point on the principal axis farther than R from the mirror surface, forms a real image at I. If the rays diverge from O at small angles, they all reflect through the same image point.

such rays reflect through the image point, as shown in Figure 36.9b. Rays that are far from the principal axis, such as those shown in Figure 36.10, converge to other points on the principal axis, producing a blurred image. This effect, which is called **spherical aberration,** is present to some extent for any spherical mirror and is discussed in Section 36.5.

We can use Figure 36.11 to calculate the image distance q from a knowledge of the object distance p and radius of curvature R. By convention, these distances are measured from point V. Figure 36.11 shows two rays leaving the tip of the object. One of these rays passes through the center of curvature C of the mirror, hitting the mirror perpendicular to the mirror surface and reflecting back on itself. The second ray strikes the mirror at its center (point V) and reflects as shown, obeying the law of reflection. The image of the tip of the arrow is located at the point where these two rays intersect. From the gold right triangle in Figure 36.11, we see that $\tan\theta = h/p$, and from the blue right triangle we see that $\tan\theta = -h'/q$. The negative sign is introduced because the image is inverted, so h' is taken to be negative. Thus, from Equation 36.1 and these results, we find that the magnification of the mirror is

$$M = \frac{h'}{h} = -\frac{q}{p} \tag{36.2}$$

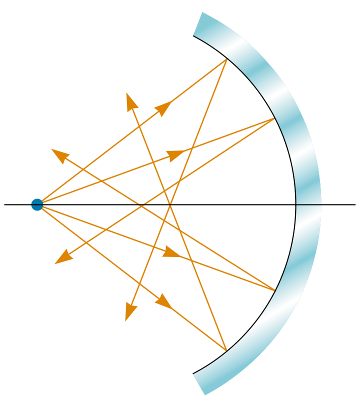


Figure 36.10 Rays diverging from the object at large angles from the principal axis reflect from a spherical concave mirror to intersect the principal axis at different points, resulting in a blurred image. This condition is called *spherical aberration*.

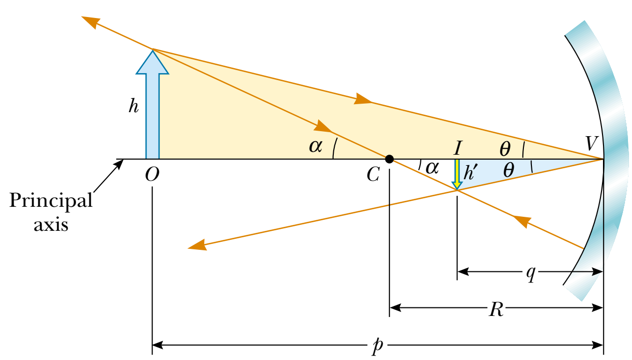


Figure 36.11 The image formed by a spherical concave mirror when the object O lies outside the center of curvature C.

We also note from the two triangles in Figure 36.11 that have α as one angle that

$$\tan \alpha = \frac{h}{p - R} \qquad \text{and} \qquad \tan \alpha = -\frac{h'}{R - q}$$

from which we find that

$$\frac{h'}{h} = -\frac{R - q}{p - R} \tag{36.3}$$

If we compare Equations 36.2 and 36.3, we see that

$$\frac{R - q}{p - R} = \frac{q}{p}$$

Simple algebra reduces this to

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R} \tag{36.4}$$

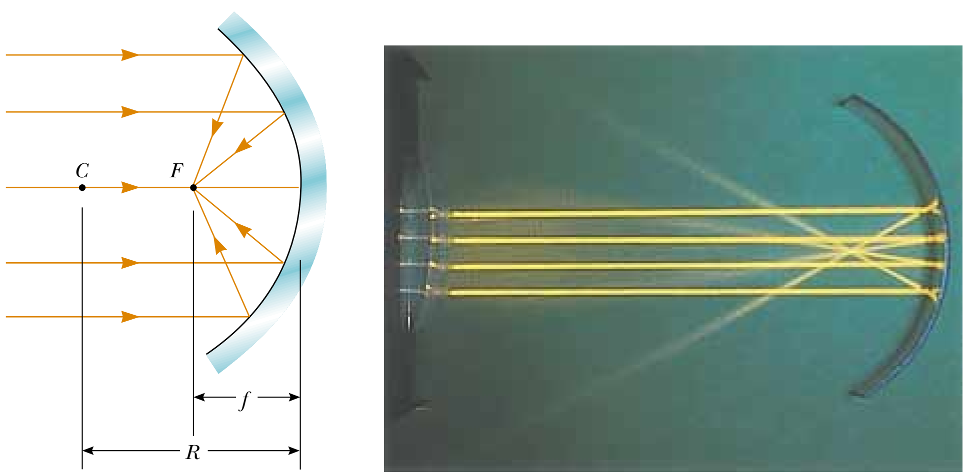
Mirror equation in terms of R

This expression is called the **mirror equation.** It is applicable only to paraxial rays.

If the object is very far from the mirror—that is, if p is so much greater than R that p can be said to approach infinity—then $1/p \approx 0$, and we see from Equation 36.4 that $q \approx R/2$. That is, when the object is very far from the mirror, the image point is halfway between the center of curvature and the center point on the mirror, as shown in Figure 36.12a. The incoming rays from the object are essentially parallel in this figure because the source is assumed to be very far from the mirror. We call the image point in this special case the **focal point** F and the image distance the **focal length** f, where

$$f = \frac{R}{2} \tag{36.5}$$

Focal length



(a) (b)

Figure 36.12 (a) Light rays from a distant object ($p \approx \infty$) reflect from a concave mirror through the focal point F. In this case, the image distance $q \approx R/2 = f$, where f is the focal length of the mirror. (b) Reflection of parallel rays from a concave mirror.

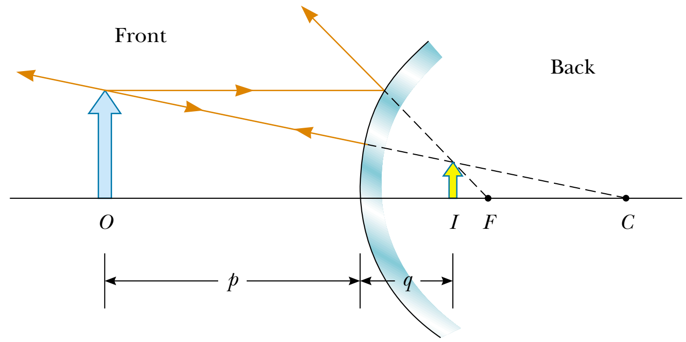


Figure 36.13 Formation of an image by a spherical convex mirror. The image formed by the real object is virtual and upright.

Focal length is a parameter particular to a given mirror and therefore can be used to compare one mirror with another. The mirror equation can be expressed in terms of the focal length:

Mirror equation in terms of f

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \tag{36.6}$$

Notice that the focal length of a mirror depends only on the curvature of the mirror and not on the material from which the mirror is made. This is because the formation of the image results from rays reflected from the surface of the material. We shall find in Section 36.4 that the situation is different for lenses; in that case the light actually passes through the material.

Convex Mirrors

Figure 36.13 shows the formation of an image by a **convex mirror**—that is, one silvered so that light is reflected from the outer, convex surface. This is sometimes called a **diverging mirror** because the rays from any point on an object diverge after reflection as though they were coming from some point behind the mirror. The image in Figure 36.13 is virtual because the reflected rays only appear to originate at the image point, as indicated by the dashed lines. Furthermore, the image is always upright and smaller than the object. This type of mirror is often used in stores to foil shoplifters. A single mirror can be used to survey a large field of view because it forms a smaller image of the interior of the store.

We do not derive any equations for convex spherical mirrors because we can use Equations 36.2, 36.4, and 36.6 for either concave or convex mirrors if we adhere to the following procedure. Let us refer to the region in which light rays move toward the mirror as the *front side* of the mirror, and the other side as the *back side*. For example, in Figures 36.10 and 36.12, the side to the left of the mirrors is the front side, and the side to the right of the mirrors is the back side. Figure 36.14 states the sign conventions for object and image distances, and Table 36.1 summarizes the sign conventions for all quantities.

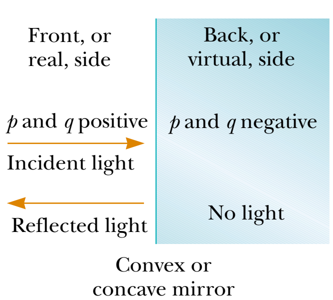


Figure 36.14 Signs of p and q for convex and concave mirrors.

TABLE 36.1 Sign Conventions for Mirrors

p is **positive** if object is in **front** of mirror (real object).
p is **negative** if object is in **back** of mirror (virtual object).

q is **positive** if image is in **front** of mirror (real image).
q is **negative** if image is in **back** of mirror (virtual image).

Both f and R are **positive** if center of curvature is in **front** of mirror (concave mirror).
Both f and R are **negative** if center of curvature is in **back** of mirror (convex mirror).

If M is **positive,** image is **upright.**
If M is **negative,** image is **inverted.**

Reflection of parallel lines from a convex cylindrical mirror. The image is virtual, upright, and reduced in size.

Ray Diagrams for Mirrors

The positions and sizes of images formed by mirrors can be conveniently determined with *ray diagrams.* These graphical constructions reveal the nature of the image and can be used to check results calculated from the mirror and magnification equations. To draw a ray diagram, we need to know the position of the object and the locations of the mirror's focal point and center of curvature. We then draw three rays to locate the image, as shown by the examples in Figure 36.15. These rays all start from the same object point and are drawn as follows. We may choose any point on the object; here, we choose the top of the object for simplicity:

- Ray 1 is drawn from the top of the object parallel to the principal axis and is reflected through the focal point F.
- Ray 2 is drawn from the top of the object through the focal point and is reflected parallel to the principal axis.
- Ray 3 is drawn from the top of the object through the center of curvature C and is reflected back on itself.

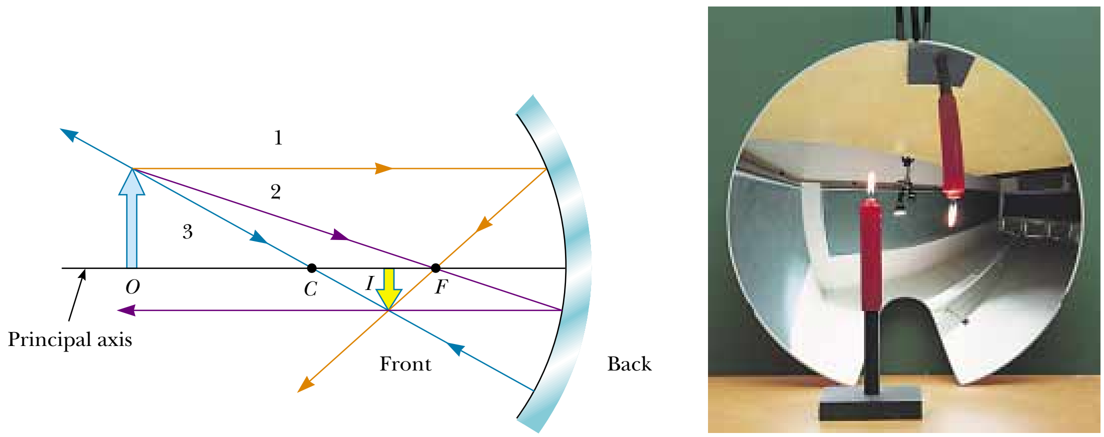
The intersection of any two of these rays locates the image. The third ray serves as a check of the construction. The image point obtained in this fashion must always agree with the value of q calculated from the mirror equation.

With concave mirrors, note what happens as the object is moved closer to the mirror. The real, inverted image in Figure 36.15a moves to the left as the object approaches the focal point. When the object is at the focal point, the image is infinitely far to the left. However, when the object lies between the focal point and the mirror surface, as shown in Figure 36.15b, the image is virtual, upright, and enlarged. This latter situation applies in the use of a shaving mirror or a makeup mirror. Your face is closer to the mirror than the focal point, and you see an upright, enlarged image of your face.

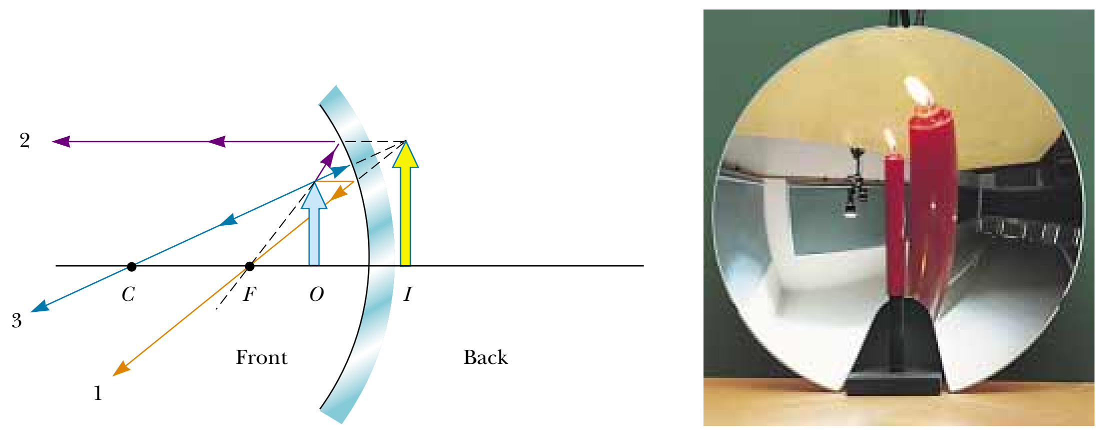
In a convex mirror (see Fig. 36.15c), the image of an object is always virtual, upright, and reduced in size. In this case, as the object distance increases, the virtual image decreases in size and approaches the focal point as p approaches infinity. You should construct other diagrams to verify how image position varies with object position.

QuickLab

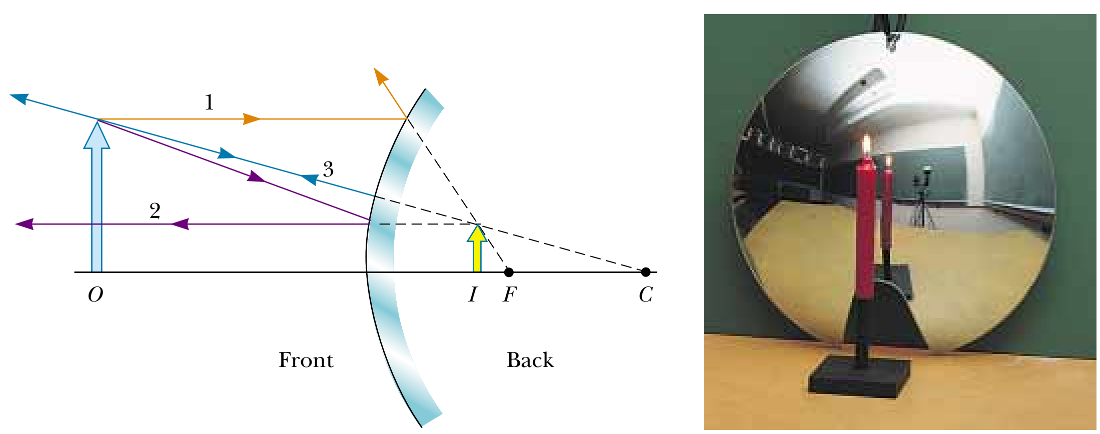
Compare the images formed of your face when you look first at the front side and then at the back side of a shiny soup spoon. Why do the two images look so different from each other?



(a)



(b)



(c)

Figure 36.15 Ray diagrams for spherical mirrors, along with corresponding photographs of the images of candles. (a) When the object is located so that the center of curvature lies between the object and a concave mirror surface, the image is real, inverted, and reduced in size. (b) When the object is located between the focal point and a concave mirror surface, the image is virtual, upright, and enlarged. (c) When the object is in front of a convex mirror, the image is virtual, upright, and reduced in size.

EXAMPLE 36.4 The Image from a Mirror

Assume that a certain spherical mirror has a focal length of $+10.0$ cm. Locate and describe the image for object distances of (a) 25.0 cm, (b) 10.0 cm, and (c) 5.00 cm.

Solution Because the focal length is positive, we know that this is a concave mirror (see Table 36.1). (a) This situation is analogous to that in Figure 36.15a; hence, we expect the image to be real and closer to the mirror than the object. According to the figure, it should also be inverted and reduced in size. We find the image distance by using the Equation 36.6 form of the mirror equation:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\frac{1}{25.0 \text{ cm}} + \frac{1}{q} = \frac{1}{10.0 \text{ cm}}$$

$$q = \boxed{16.7 \text{ cm}}$$

The magnification is given by Equation 36.2:

$$M = -\frac{q}{p} = -\frac{16.7 \text{ cm}}{25.0 \text{ cm}} = -0.668$$

The fact that the absolute value of M is less than unity tells us that the image is smaller than the object, and the negative sign for M tells us that the image is inverted. Because q is positive, the image is located on the front side of the mirror and is real. Thus, we see that our predictions were correct.

(b) When the object distance is 10.0 cm, the object is located at the focal point. Now we find that

$$\frac{1}{10.0 \text{ cm}} + \frac{1}{q} = \frac{1}{10.0 \text{ cm}}$$

$$q = \boxed{\infty}$$

which means that rays originating from an object positioned at the focal point of a mirror are reflected so that the image is formed at an infinite distance from the mirror; that is, the rays travel parallel to one another after reflection. This is the situation in a flashlight, where the bulb filament is placed at the focal point of a reflector, producing a parallel beam of light.

(c) When the object is at $p = 5.00$ cm, it lies between the focal point and the mirror surface, as shown in Figure 36.15b. Thus, we expect a magnified, virtual, upright image. In this case, the mirror equation gives

$$\frac{1}{5.00 \text{ cm}} + \frac{1}{q} = \frac{1}{10.0 \text{ cm}}$$

$$q = \boxed{-10.0 \text{ cm}}$$

The image is virtual because it is located behind the mirror, as expected. The magnification is

$$M = -\frac{q}{p} = -\left(\frac{-10.0 \text{ cm}}{5.00 \text{ cm}}\right) = 2.00$$

The image is twice as large as the object, and the positive sign for M indicates that the image is upright (see Fig. 36.15b).

Exercise At what object distance is the magnification -1.00?

Answer 20.0 cm.

EXAMPLE 36.5 The Image from a Convex Mirror

A woman who is 1.5 m tall is located 3.0 m from an antishoplifting mirror, as shown in Figure 36.16. The focal length of the mirror is -0.25 m. Find (a) the position of her image and (b) the magnification.

Solution (a) This situation is depicted in Figure 36.15c. We should expect to find an upright, reduced, virtual image. To find the image position, we use Equation 36.6:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} = \frac{1}{-0.25 \text{ m}}$$

$$\frac{1}{q} = \frac{1}{-0.25 \text{ m}} - \frac{1}{3.0 \text{ m}}$$

$$q = \boxed{-0.23 \text{ m}}$$

Figure 36.16 Convex mirrors, often used for security in department stores, provide wide-angle viewing.

The negative value of q indicates that her image is virtual, or behind the mirror, as shown in Figure 36.15c.

(b) The magnification is

$$M = -\frac{q}{p} = -\left(\frac{-0.23 \text{ m}}{3.0 \text{ m}}\right) = \boxed{0.077}$$

The image is much smaller than the woman, and it is upright because M is positive.

Exercise Find the height of the image.

Answer 0.12 m.

36.3 IMAGES FORMED BY REFRACTION

In this section we describe how images are formed when light rays are refracted at the boundary between two transparent materials. Consider two transparent media having indices of refraction n_1 and n_2, where the boundary between the two media is a spherical surface of radius R (Fig. 36.17). We assume that the object at O is in the medium for which the index of refraction is n_1, where $n_1 < n_2$. Let us consider the paraxial rays leaving O. As we shall see, all such rays are refracted at the spherical surface and focus at a single point I, the image point.

Figure 36.18 shows a single ray leaving point O and focusing at point I. Snell's law of refraction applied to this refracted ray gives

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

Because θ_1 and θ_2 are assumed to be small, we can use the small-angle approximation $\sin \theta \approx \theta$ (angles in radians) and say that

$$n_1\theta_1 = n_2\theta_2$$

Now we use the fact that an exterior angle of any triangle equals the sum of the two opposite interior angles. Applying this rule to triangles OPC and PIC in Figure 36.18 gives

$$\theta_1 = \alpha + \beta$$

$$\beta = \theta_2 + \gamma$$

If we combine all three expressions and eliminate θ_1 and θ_2, we find that

$$n_1\alpha + n_2\gamma = (n_2 - n_1)\beta \qquad \textbf{(36.7)}$$

Looking at Figure 36.18, we see three right triangles that have a common vertical leg of length d. For paraxial rays (unlike the relatively large-angle ray shown in Fig.

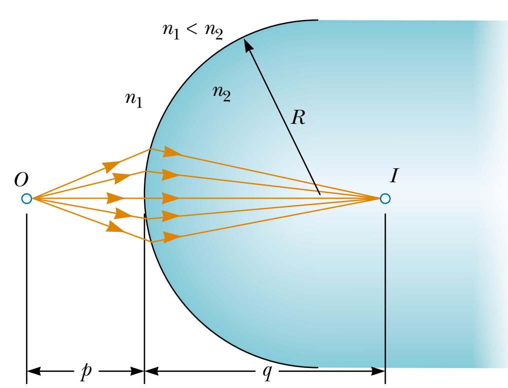


Figure 36.17 An image formed by refraction at a spherical surface. Rays making small angles with the principal axis diverge from a point object at O and are refracted through the image point I.

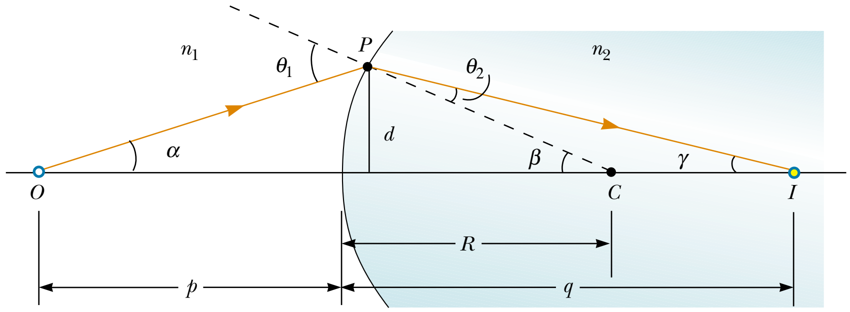


Figure 36.18 Geometry used to derive Equation 36.8.

36.18), the horizontal legs of these triangles are approximately p for the triangle containing angle α, R for the triangle containing angle β, and q for the triangle containing angle γ. In the small-angle approximation, $\tan\theta \approx \theta$, so we can write the approximate relationships from these triangles as follows:

$$\tan\alpha \approx \alpha \approx \frac{d}{p} \qquad \tan\beta \approx \beta \approx \frac{d}{R} \qquad \tan\gamma \approx \gamma \approx \frac{d}{q}$$

We substitute these expressions into Equation 36.7 and divide through by d to get

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \tag{36.8}$$

For a fixed object distance p, the image distance q is independent of the angle that the ray makes with the axis. This result tells us that all paraxial rays focus at the same point I.

As with mirrors, we must use a sign convention if we are to apply this equation to a variety of cases. We define the side of the surface in which light rays originate as the front side. The other side is called the back side. Real images are formed by refraction in back of the surface, in contrast with mirrors, where real images are formed in front of the reflecting surface. Because of the difference in location of real images, the refraction sign conventions for q and R are opposite the reflection sign conventions. For example, q and R are both positive in Figure 36.18. The sign conventions for spherical refracting surfaces are summarized in Table 36.2.

We derived Equation 36.8 from an assumption that $n_1 < n_2$. This assumption is not necessary, however. Equation 36.8 is valid regardless of which index of refraction is greater.

TABLE 36.2 Sign Conventions for Refracting Surfaces

p is **positive** if object is in **front** of surface (real object).
p is **negative** if object is in **back** of surface (virtual object).

q is **positive** if image is in **back** of surface (real image).
q is **negative** if image is in **front** of surface (virtual image).

R is **positive** if center of curvature is in **back** of convex surface.
R is **negative** if center of curvature is in **front** of concave surface.

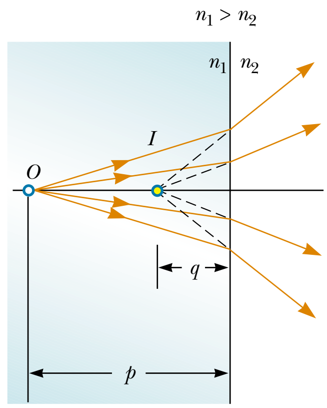


Figure 36.19 The image formed by a flat refracting surface is virtual and on the same side of the surface as the object. All rays are assumed to be paraxial.

Flat Refracting Surfaces

If a refracting surface is flat, then R is infinite and Equation 36.8 reduces to

$$\frac{n_1}{p} = -\frac{n_2}{q}$$

$$q = -\frac{n_2}{n_1}p \qquad \textbf{(36.9)}$$

From this expression we see that the sign of q is opposite that of p. Thus, according to Table 36.2, **the image formed by a flat refracting surface is on the same side of the surface as the object.** This is illustrated in Figure 36.19 for the situation in which the object is in the medium of index n_1 and n_1 is greater than n_2. In this case, a virtual image is formed between the object and the surface. If n_1 is less than n_2, the rays in the back side diverge from each other at lesser angles than those in Figure 36.19. As a result, the virtual image is formed to the left of the object.

CONCEPTUAL EXAMPLE 36.6 Let's Go Scuba Diving!

It is well known that objects viewed under water with the naked eye appear blurred and out of focus. However, a scuba diver using a mask has a clear view of underwater objects. (a) Explain how this works, using the facts that the indices of refraction of the cornea, water, and air are 1.376, 1.333, and 1.000 29, respectively.

Solution Because the cornea and water have almost identical indices of refraction, very little refraction occurs when a person under water views objects with the naked eye. In this case, light rays from an object focus behind the retina, resulting in a blurred image. When a mask is used, the air space between the eye and the mask surface provides the normal amount of refraction at the eye–air interface, and the light from the object is focused on the retina.

(b) If a lens prescription is ground into the glass of a mask, should the curved surface be on the inside of the mask, the outside, or both?

Solution If a lens prescription is ground into the glass of the mask so that the wearer can see without eyeglasses, only the inside surface is curved. In this way the prescription is accurate whether the mask is used under water or in air. If the curvature were on the outer surface, the refraction at the outer surface of the glass would change depending on whether air or water were present on the outside of the mask.

EXAMPLE 36.7 Gaze into the Crystal Ball

A dandelion seed ball 4.0 cm in diameter is embedded in the center of a spherical plastic paperweight having a diameter of 6.0 cm (Fig. 36.20a). The index of refraction of the plastic is $n_1 = 1.50$. Find the position of the image of the near edge of the seed ball.

Solution Because $n_1 > n_2$, where $n_2 = 1.00$ is the index of refraction for air, the rays originating from the seed ball are refracted away from the normal at the surface and diverge outward, as shown in Figure 36.20b. Hence, the image is formed inside the paperweight and is virtual. From the given dimensions, we know that the near edge of the seed ball is 1.0 cm beneath the surface of the paperweight. Applying Equation 36.8 and noting from Table 36.2 that R is negative, we obtain

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R}$$

$$\frac{1.50}{1.0\text{ cm}} + \frac{1}{q} = \frac{1.00 - 1.50}{-3.0\text{ cm}}$$

$$q = \boxed{-0.75\text{ cm}}$$

The negative sign for q indicates that the image is in front of the surface—in other words, in the same medium as the object, as shown in Figure 36.20b. Being in the same medium as the object, the image must be virtual (see Table 36.2). The surface of the seed ball appears to be closer to the paperweight surface than it actually is.

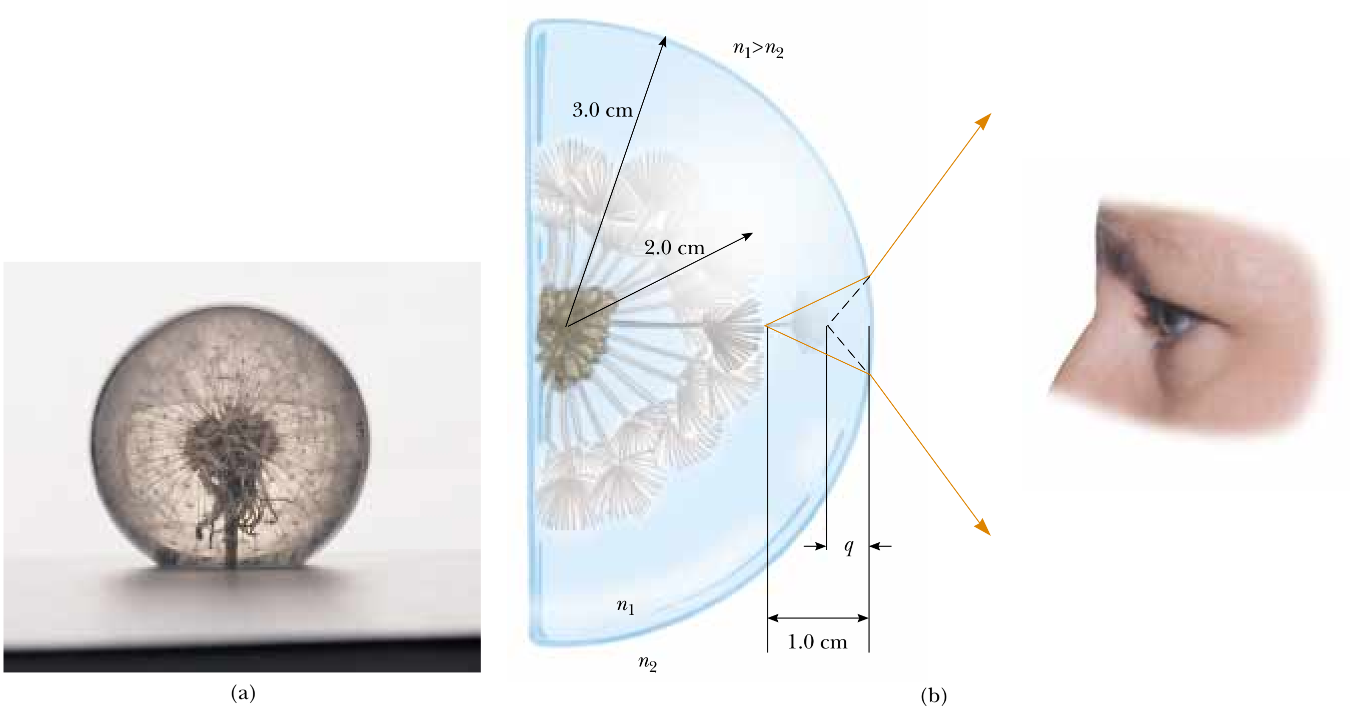


Figure 36.20 (a) An object embedded in a plastic sphere forms a virtual image between the surface of the object and the sphere surface. All rays are assumed paraxial. Because the object is inside the sphere, the front of the refracting surface is the *interior* of the sphere. (b) Rays from the surface of the object form an image that is still inside the plastic sphere but closer to the plastic surface.

EXAMPLE 36.8 The One That Got Away

A small fish is swimming at a depth d below the surface of a pond (Fig. 36.21). What is the apparent depth of the fish, as viewed from directly overhead?

Solution Because the refracting surface is flat, R is infinite. Hence, we can use Equation 36.9 to determine the location of the image with $p = d$. Using the indices of refraction given in Figure 36.21, we obtain

$$q = -\frac{n_2}{n_1}p = -\frac{1.00}{1.33}d = \boxed{-0.752d}$$

Because q is negative, the image is virtual, as indicated by the dashed lines in Figure 36.21. The apparent depth is three-fourths the actual depth.

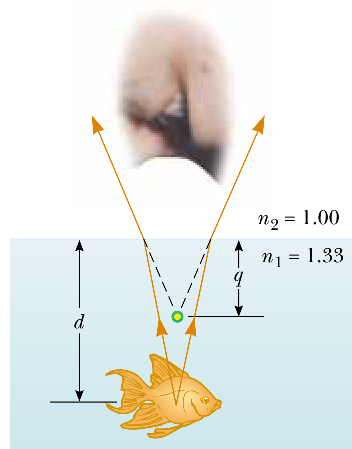


Figure 36.21 The apparent depth q of the fish is less than the true depth d. All rays are assumed to be paraxial.

36.4 THIN LENSES

14.8 Lenses are commonly used to form images by refraction in optical instruments, such as cameras, telescopes, and microscopes. We can use what we just learned about images formed by refracting surfaces to help us locate the image formed by a lens. We recognize that light passing through a lens experiences refraction at two surfaces. The development we shall follow is based on the notion that **the image formed by one refracting surface serves as the object for the second surface.** We shall analyze a thick lens first and then let the thickness of the lens be approximately zero.

Consider a lens having an index of refraction n and two spherical surfaces with radii of curvature R_1 and R_2, as in Figure 36.22. (Note that R_1 is the radius of curvature of the lens surface that the light leaving the object reaches first and that R_2 is the radius of curvature of the other surface of the lens.) An object is placed at point O at a distance p_1 in front of surface 1. If the object were far from surface 1, the light rays from the object that struck the surface would be almost parallel to each other. The refraction at the surface would focus these rays, forming a real image to the right of surface 1 in Figure 36.22 (as in Fig. 36.17). If the object is placed close to surface 1, as shown in Figure 36.22, the rays diverging from the object and striking the surface cover a wide range of angles and are not parallel to each other. In this case, the refraction at the surface is not sufficient to cause the rays to converge on the right side of the surface. They still diverge, although they are closer to parallel than they were before they struck the surface. This results in a virtual image of the object at I_1 to the left of the surface, as shown in Figure 36.22. This image is then used as the object for surface 2, which results in a real image I_2 to the right of the lens.

Let us begin with the virtual image formed by surface 1. Using Equation 36.8 and assuming that $n_1 = 1$ because the lens is surrounded by air, we find that the image I_1 formed by surface 1 satisfies the equation

$$(1) \qquad \frac{1}{p_1} + \frac{n}{q_1} = \frac{n-1}{R_1}$$

where q_1 is a negative number because it represents a virtual image formed on the front side of surface 1.

Now we apply Equation 36.8 to surface 2, taking $n_1 = n$ and $n_2 = 1$. (We make this switch in index because the light rays from I_1 approaching surface 2 are *in the material of the lens,* and this material has index n. We could also imagine removing the object at O, filling all of the space to the left of surface 1 with the mate-

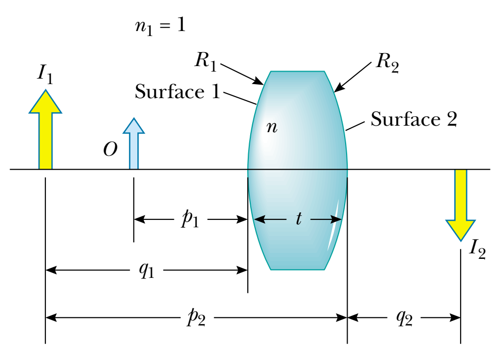


Figure 36.22 To locate the image formed by a lens, we use the virtual image at I_1 formed by surface 1 as the object for the image formed by surface 2. The final image is real and is located at I_2.

rial of the lens, and placing the object at I_1; the light rays approaching surface 2 would be the same as in the actual situation in Fig. 36.22.) Taking p_2 as the object distance for surface 2 and q_2 as the image distance gives

$$(2) \qquad \frac{n}{p_2} + \frac{1}{q_2} = \frac{1-n}{R_2}$$

We now introduce mathematically the fact that the image formed by the first surface acts as the object for the second surface. We do this by noting from Figure 36.22 that p_2 is the sum of q_1 and t and by setting $p_2 = -q_1 + t$, where t is the thickness of the lens. (Remember that q_1 is a negative number and that p_2 must be positive by our sign convention—thus, we must introduce a negative sign for q_1.) For a *thin* lens (for which the thickness is small compared to the radii of curvature), we can neglect t. In this approximation, we see that $p_2 = -q_1$. Hence, Equation (2) becomes

$$(3) \qquad -\frac{n}{q_1} + \frac{1}{q_2} = \frac{1-n}{R_2}$$

Adding Equations (1) and (3), we find that

$$(4) \qquad \frac{1}{p_1} + \frac{1}{q_2} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

For a thin lens, we can omit the subscripts on p_1 and q_2 in Equation (4) and call the object distance p and the image distance q, as in Figure 36.23. Hence, we can write Equation (4) in the form

$$\frac{1}{p} + \frac{1}{q} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{36.10}$$

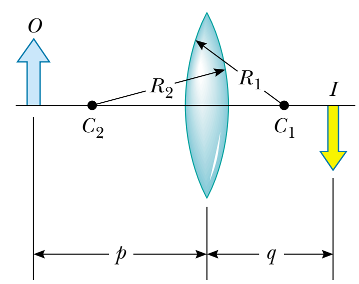


Figure 36.23 Simplified geometry for a thin lens.

This expression relates the image distance q of the image formed by a thin lens to the object distance p and to the thin-lens properties (index of refraction and radii of curvature). It is valid only for paraxial rays and only when the lens thickness is much less than R_1 and R_2.

The **focal length** f of a thin lens is the image distance that corresponds to an infinite object distance, just as with mirrors. Letting p approach ∞ and q approach f in Equation 36.10, we see that the inverse of the focal length for a thin lens is

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{36.11}$$

Lens makers' equation

This relationship is called the **lens makers' equation** because it can be used to determine the values of R_1 and R_2 that are needed for a given index of refraction and a desired focal length f. Conversely, if the index of refraction and the radii of curvature of a lens are given, this equation enables a calculation of the focal length. If the lens is immersed in something other than air, this same equation can be used, with n interpreted as the *ratio* of the index of refraction of the lens material to that of the surrounding fluid.

Quick Quiz 36.2

What is the focal length of a pane of window glass?

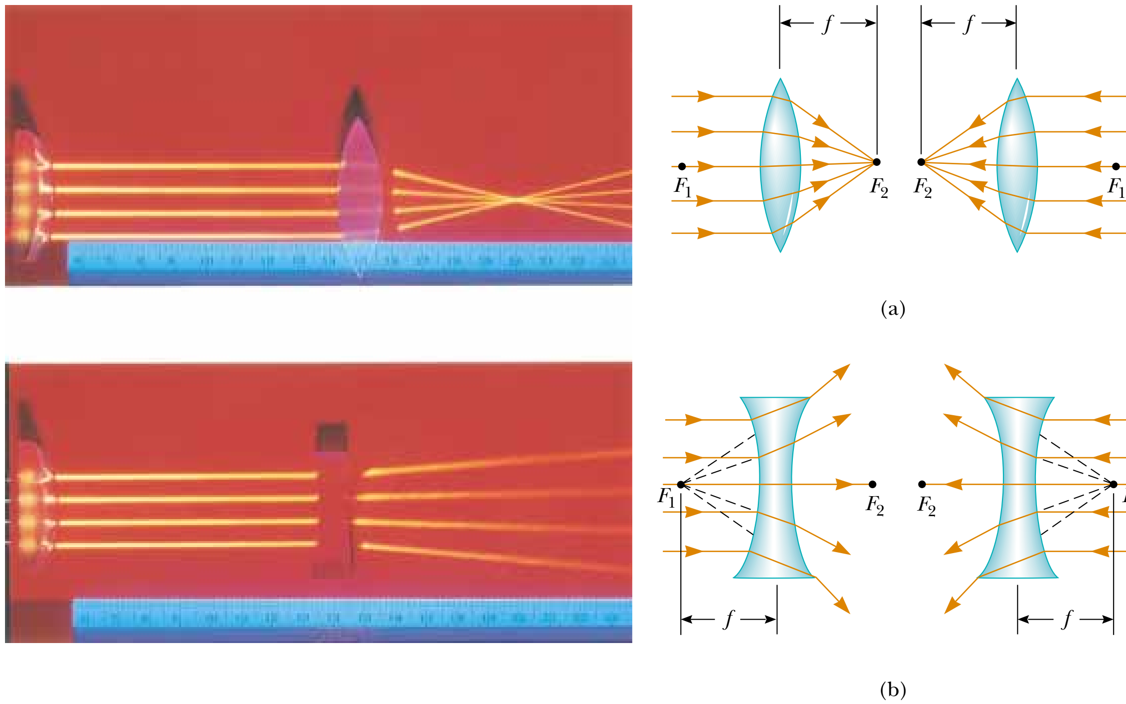


Figure 36.24 (*Left*) Effects of a converging lens (top) and a diverging lens (bottom) on parallel rays. (*Right*) The object and image focal points of (a) a converging lens and (b) a diverging lens.

Using Equation 36.11, we can write Equation 36.10 in a form identical to Equation 36.6 for mirrors:

Thin-lens equation

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \tag{36.12}$$

This equation, called the **thin-lens equation,** can be used to relate the image distance and object distance for a thin lens.

Because light can travel in either direction through a lens, each lens has two focal points, one for light rays passing through in one direction and one for rays passing through in the other direction. This is illustrated in Figure 36.24 for a biconvex lens (two convex surfaces, resulting in a converging lens) and a biconcave lens (two concave surfaces, resulting in a diverging lens). Focal point F_1 is sometimes called the *object focal point,* and F_2 is called the *image focal point.*

Figure 36.25 is useful for obtaining the signs of p and q, and Table 36.3 gives the sign conventions for thin lenses. Note that these sign conventions are the same as those for refracting surfaces (see Table 36.2). Applying these rules to a biconvex lens, we see that when $p > f$, the quantities p, q, and R_1 are positive, and R_2 is negative. Therefore, p, q, and f are all positive when a converging lens forms a real image of an object. For a biconcave lens, p and R_2 are positive and q and R_1 are negative, with the result that f is negative.

Various lens shapes are shown in Figure 36.26. Note that a converging lens is thicker at the center than at the edge, whereas a diverging lens is thinner at the center than at the edge.

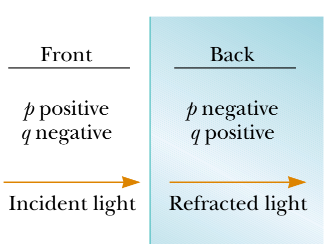


Figure 36.25 A diagram for obtaining the signs of p and q for a thin lens. (This diagram also applies to a refracting surface.)

TABLE 36.3 Sign Conventions for Thin Lenses

p is **positive** if object is in **front** of lens (real object).
p is **negative** if object is in **back** of lens (virtual object).

q is **positive** if image is in **back** of lens (real image).
q is **negative** if image is in **front** of lens (virtual image).

R_1 and R_2 are **positive** if center of curvature is in **back** of lens.
R_1 and R_2 are **negative** if center of curvature is in **front** of lens.

f is **positive** if the lens is **converging.**
f is **negative** if the lens is **diverging.**

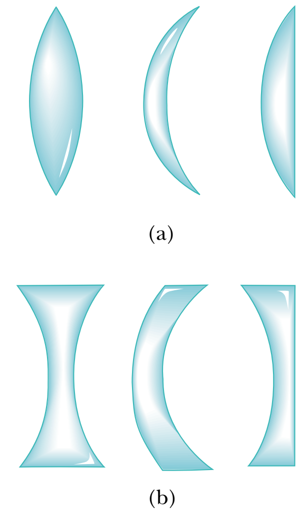


Figure 36.26 Various lens shapes. (a) Biconvex, convex–concave, and plano–convex. These are all converging lenses; they have a positive focal length and are thickest at the middle. (b) Biconcave, convex–concave, and plano–concave. These are all diverging lenses; they have a negative focal length and are thickest at the edges.

Magnification of Images

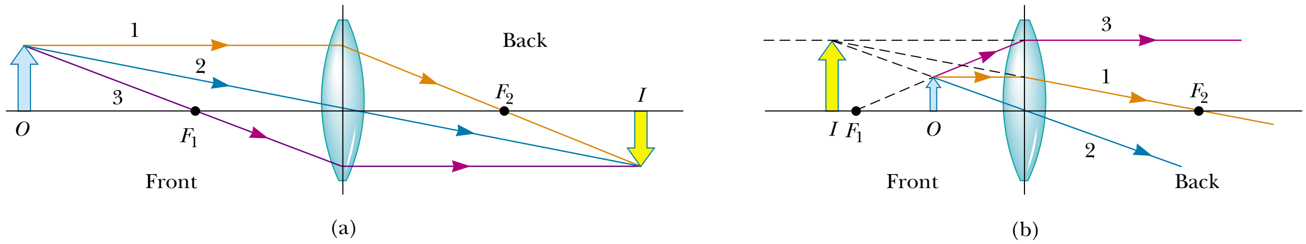
Consider a thin lens through which light rays from an object pass. As with mirrors (Eq. 36.2), the lateral magnification of the lens is defined as the ratio of the image height h' to the object height h:

$$M = \frac{h'}{h} = -\frac{q}{p}$$

From this expression, it follows that when M is positive, the image is upright and on the same side of the lens as the object. When M is negative, the image is inverted and on the side of the lens opposite the object.

Ray Diagrams for Thin Lenses

Ray diagrams are convenient for locating the images formed by thin lenses or systems of lenses. They also help clarify our sign conventions. Figure 36.27 shows such diagrams for three single-lens situations. To locate the image of a converg-



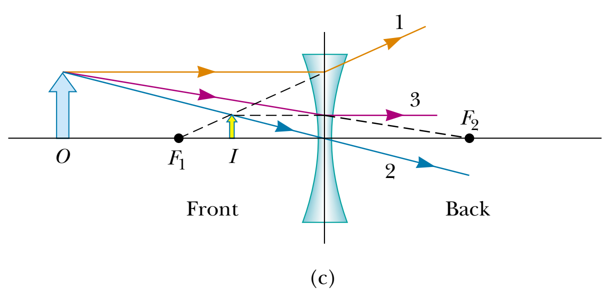


Figure 36.27 Ray diagrams for locating the image formed by a thin lens. (a) When the object is in front of and outside the object focal point F_1 of a converging lens, the image is real, inverted, and on the back side of the lens. (b) When the object is between F_1 and a converging lens, the image is virtual, upright, larger than the object, and on the front side of the lens. (c) When an object is anywhere in front of a diverging lens, the image is virtual, upright, smaller than the object, and on the front side of the lens.

ing lens (Fig. 36.27a and b), the following three rays are drawn from the top of the object:

- Ray 1 is drawn parallel to the principal axis. After being refracted by the lens, this ray passes through the focal point on the back side of the lens.
- Ray 2 is drawn through the center of the lens and continues in a straight line.
- Ray 3 is drawn through that focal point on the front side of the lens (or as if coming from the focal point if $p < f$) and emerges from the lens parallel to the principal axis.

To locate the image of a diverging lens (Fig. 36.27c), the following three rays are drawn from the top of the object:

- Ray 1 is drawn parallel to the principal axis. After being refracted by the lens, this ray emerges such that it appears to have passed through the focal point on the front side of the lens. (This apparent direction is indicated by the dashed line in Fig. 36.27c.)
- Ray 2 is drawn through the center of the lens and continues in a straight line.
- Ray 3 is drawn toward the focal point on the back side of the lens and emerges from the lens parallel to the optic axis.

Quick Quiz 36.3

In Figure 36.27a, the blue object arrow is replaced by one that is much taller than the lens. How many rays from the object will strike the lens?

For the converging lens in Figure 36.27a, where the object is to the left of the object focal point ($p > f_1$), the image is real and inverted. When the object is between the object focal point and the lens ($p < f_1$), as shown in Figure 36.27b, the image is virtual and upright. For a diverging lens (see Fig. 36.27c), the image is always virtual and upright, regardless of where the object is placed. These geometric constructions are reasonably accurate only if the distance between the rays and the principal axis is much less than the radii of the lens surfaces.

It is important to realize that refraction occurs only at the surfaces of the lens. A certain lens design takes advantage of this fact to produce the *Fresnel lens,* a powerful lens without great thickness. Because only the surface curvature is important in the refracting qualities of the lens, material in the middle of a Fresnel lens is removed, as shown in Figure 36.28. Because the edges of the curved segments cause some distortion, Fresnel lenses are usually used only in situations in which image quality is less important than reduction of weight.

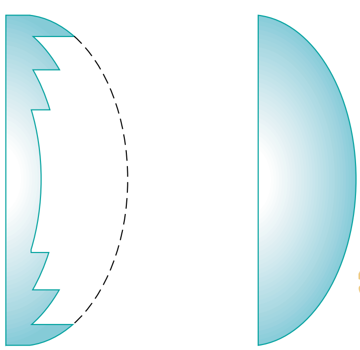


Figure 36.28 The Fresnel lens on the left has the same focal length as the thick lens on the right but is made of much less glass.

The lines that are visible across the faces of most automobile headlights are the edges of these curved segments. A headlight requires a short-focal-length lens to collimate light from the nearby filament into a parallel beam. If it were not for the Fresnel design, the glass would be very thick in the center and quite heavy. The weight of the glass would probably cause the thin edge where the lens is supported to break when subjected to the shocks and vibrations that are typical of travel on rough roads.

Quick Quiz 36.4

If you cover the top half of a lens, which of the following happens to the appearance of the image of an object? (a) The bottom half disappears; (b) the top half disappears; (c) the entire image is visible but has half the intensity; (d) no change occurs; (e) the entire image disappears.

Example 36.9 An Image Formed by a Diverging Lens

A diverging lens has a focal length of -20.0 cm. An object 2.00 cm tall is placed 30.0 cm in front of the lens. Locate the image.

Solution Using the thin-lens equation (Eq. 36.12) with $p = 30.0$ cm and $f = -20.0$ cm, we obtain

$$\frac{1}{30.0\text{ cm}} + \frac{1}{q} = \frac{1}{-20.0\text{ cm}}$$

$$q = \boxed{-12.0\text{ cm}}$$

The negative sign tells us that the image is in front of the lens and virtual, as indicated in Figure 36.27c.

Exercise Determine both the magnification and the height of the image.

Answer $M = 0.400$; $h' = 0.800$ cm.

Example 36.10 An Image Formed by a Converging Lens

A converging lens of focal length 10.0 cm forms an image of each of three objects placed (a) 30.0 cm, (b) 10.0 cm, and (c) 5.00 cm in front of the lens. In each case, find the image distance and describe the image.

Solution (a) The thin-lens equation can be used again:

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f}$$

$$\frac{1}{30.0\text{ cm}} + \frac{1}{q} = \frac{1}{10.0\text{ cm}}$$

$$q = \boxed{15.0\text{ cm}}$$

The positive sign indicates that the image is in back of the lens and real. The magnification is

$$M = -\frac{q}{p} = -\frac{15.0\text{ cm}}{30.0\text{ cm}} = \boxed{-0.500}$$

The image is reduced in size by one half, and the negative sign for M means that the image is inverted. The situation is like that pictured in Figure 36.27a.

(b) No calculation is necessary for this case because we know that, when the object is placed at the focal point, the image is formed at infinity. We can readily verify this by substituting $p = 10.0$ cm into the thin-lens equation.

(c) We now move inside the focal point, to an object distance of 5.00 cm:

$$\frac{1}{5.00\text{ cm}} + \frac{1}{q} = \frac{1}{10.0\text{ cm}}$$

$$q = \boxed{-10.0\text{ cm}}$$

$$M = -\frac{q}{p} = -\left(\frac{-10.0\text{ cm}}{5.00\text{ cm}}\right) = \boxed{2.00}$$

The negative image distance indicates that the image is in front of the lens and virtual. The image is enlarged, and the positive sign for M tells us that the image is upright, as shown in Figure 36.27b.

Example 36.11 A Lens Under Water

A converging glass lens ($n = 1.52$) has a focal length of 40.0 cm in air. Find its focal length when it is immersed in water, which has an index of refraction of 1.33.

Solution We can use the lens makers' equation (Eq. 36.11) in both cases, noting that R_1 and R_2 remain the same in air and water:

$$\frac{1}{f_{\text{air}}} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

$$\frac{1}{f_{\text{water}}} = (n'-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where n' is the ratio of the index of refraction of glass to that of water: $n' = 1.52/1.33 = 1.14$. Dividing the first equation by the second gives

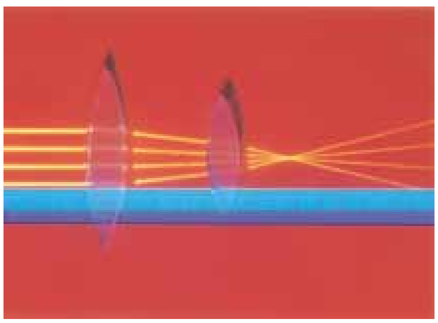
$$\frac{f_{\text{water}}}{f_{\text{air}}} = \frac{n-1}{n'-1} = \frac{1.52-1}{1.14-1} = 3.71$$

Because $f_{\text{air}} = 40.0$ cm, we find that

$$f_{\text{water}} = 3.71 f_{\text{air}} = 3.71(40.0 \text{ cm}) = \boxed{148 \text{ cm}}$$

The focal length of any glass lens is increased by a factor $(n-1)/(n'-1)$ when the lens is immersed in water.

Combination of Thin Lenses



Light from a distant object brought into focus by two converging lenses.

If two thin lenses are used to form an image, the system can be treated in the following manner. First, the image formed by the first lens is located as if the second lens were not present. Then a ray diagram is drawn for the second lens, with the image formed by the first lens now serving as the object for the second lens. The second image formed is the final image of the system. One configuration is particularly straightforward; that is, if the image formed by the first lens lies on the back side of the second lens, then that image is treated as a **virtual object** for the second lens (that is, p is negative). The same procedure can be extended to a system of three or more lenses. The overall magnification of a system of thin lenses equals the product of the magnifications of the separate lenses.

Let us consider the special case of a system of two lenses in contact. Suppose two thin lenses of focal lengths f_1 and f_2 are placed in contact with each other. If p is the object distance for the combination, application of the thin-lens equation (Eq. 36.12) to the first lens gives

$$\frac{1}{p} + \frac{1}{q_1} = \frac{1}{f_1}$$

where q_1 is the image distance for the first lens. Treating this image as the object for the second lens, we see that the object distance for the second lens must be $-q_1$ (negative because the object is virtual). Therefore, for the second lens,

$$\frac{1}{-q_1} + \frac{1}{q} = \frac{1}{f_2}$$

where q is the final image distance from the second lens. Adding these equations eliminates q_1 and gives

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f_1} + \frac{1}{f_2}$$

Focal length of two thin lenses in contact

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} \qquad \textbf{(36.13)}$$

Because the two thin lenses are touching, q is also the distance of the final image from the first lens. Therefore, **two thin lenses in contact with each other are equivalent to a single thin lens having a focal length given by Equation 36.13.**

EXAMPLE 36.12 Where Is the Final Image?

Even when the conditions just described do not apply, the lens equations yield image position and magnification. For example, two thin converging lenses of focal lengths $f_1 = 10.0$ cm and $f_2 = 20.0$ cm are separated by 20.0 cm, as illustrated in Figure 36.29. An object is placed 15.0 cm to the left of lens 1. Find the position of the final image and the magnification of the system.

Solution First we locate the image formed by lens 1 while ignoring lens 2:

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{f_1}$$

$$\frac{1}{15.0 \text{ cm}} + \frac{1}{q_1} = \frac{1}{10.0 \text{ cm}}$$

$$q_1 = 30.0 \text{ cm}$$

where q_1 is measured from lens 1. A positive value for q_1 means that this first image is in back of lens 1.

Because q_1 is greater than the separation between the two lenses, this image formed by lens 1 lies 10.0 cm to the right of lens 2. We take this as the object distance for the second lens, so $p_2 = -10.0$ cm, where distances are now measured from lens 2:

$$\frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{f_2}$$

$$\frac{1}{-10.0 \text{ cm}} + \frac{1}{q_2} = \frac{1}{20.0 \text{ cm}}$$

$$q_2 = \boxed{6.67 \text{ cm}}$$

The final image lies 6.67 cm to the right of lens 2.

The individual magnifications of the images are

$$M_1 = -\frac{q_1}{p_1} = -\frac{30.0 \text{ cm}}{15.0 \text{ cm}} = -2.00$$

$$M_2 = -\frac{q_2}{p_2} = -\frac{6.67 \text{ cm}}{-10.0 \text{ cm}} = 0.667$$

The total magnification M is equal to the product $M_1M_2 = (-2.00)(0.667) = \boxed{-1.33.}$ The final image is real because q_2 is positive. The image is also inverted and enlarged.

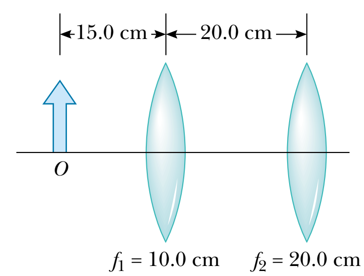


Figure 36.29 A combination of two converging lenses.

CONCEPTUAL EXAMPLE 36.13 Watch Your *p*'s and *q*'s!

Use a spreadsheet or a similar tool to create two graphs of image distance as a function of object distance—one for a lens for which the focal length is 10 cm and one for a lens for which the focal length is -10 cm.

Solution The graphs are shown in Figure 36.30. In each graph a gap occurs where $p = f$, which we shall discuss. Note the similarity in the shapes—a result of the fact that image and object distances for both lenses are related according to the same equation—the thin-lens equation.

The curve in the upper right portion of the $f = +10$ cm graph corresponds to an object on the *front* side of a lens, which we have drawn as the left side of the lens in our previous diagrams. When the object is at positive infinity, a real image forms at the focal point on the back side (the positive side) of the lens, $q = f$. (The incoming rays are parallel in this case.) As the object gets closer to the lens, the image moves farther from the lens, corresponding to the upward path of the curve. This continues until the object is located at the focal point on the near side of the lens. At this point, the rays leaving the lens are parallel, making the image infinitely far away. This is described in the graph by the asymptotic approach of the curve to the line $p = f = 10$ cm.

As the object moves inside the focal point, the image becomes virtual and located near $q = -\infty$. We are now following the curve in the lower left portion of Figure 36.30a. As the object moves closer to the lens, the virtual image also moves closer to the lens. As $p \rightarrow 0$, the image distance q also approaches 0. Now imagine that we bring the object to the back side of the lens, where $p < 0$. The object is now a virtual object, so it must have been formed by some other lens. For all locations of the virtual object, the image distance is positive and less than the focal length. The final image is real, and its position approaches the focal point as p gets more and more negative.

The $f = -10$ cm graph shows that a distant real object forms an image at the focal point on the front side of the lens. As the object approaches the lens, the image remains

virtual and moves closer to the lens. But as we continue toward the left end of the p axis, the object becomes virtual. As the position of this virtual object approaches the focal point, the image recedes toward infinity. As we pass the focal point, the image shifts from a location at positive infinity to one at negative infinity. Finally, as the virtual object continues moving away from the lens, the image is virtual, starts moving in from negative infinity, and approaches the focal point.

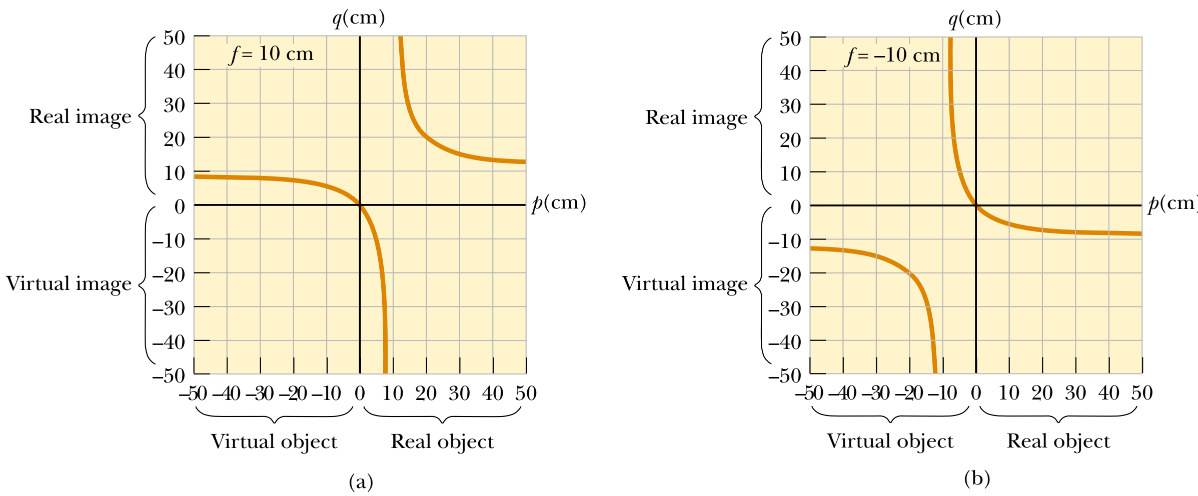


Figure 36.30 (a) Image position as a function of object position for a lens having a focal length of +10 cm. (b) Image position as a function of object position for a lens having a focal length of −10 cm.

Optional Section

36.5 LENS ABERRATIONS

One problem with lenses is imperfect images. The theory of mirrors and lenses that we have been using assumes that rays make small angles with the principal axis and that the lenses are thin. In this simple model, all rays leaving a point source focus at a single point, producing a sharp image. Clearly, this is not always true. When the approximations used in this theory do not hold, imperfect images are formed.

A precise analysis of image formation requires tracing each ray, using Snell's law at each refracting surface and the law of reflection at each reflecting surface. This procedure shows that the rays from a point object do not focus at a single point, with the result that the image is blurred. The departures of actual (imperfect) images from the ideal predicted by theory are called **aberrations.**

Spherical Aberrations

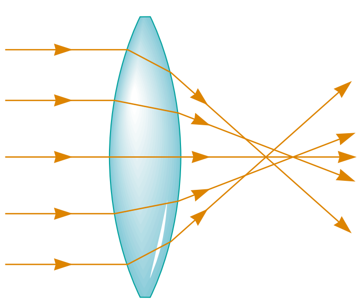
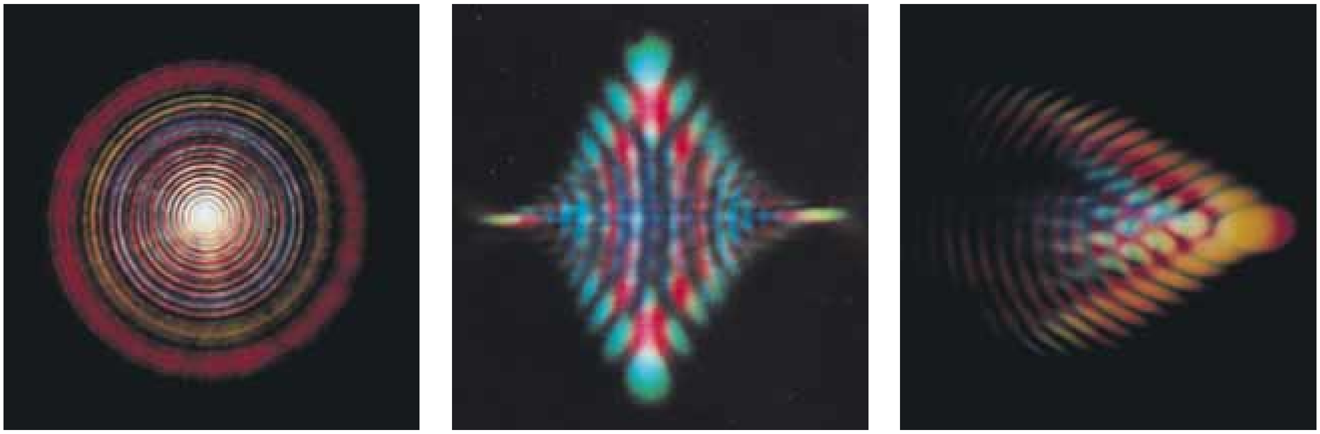


Figure 36.31 Spherical aberration caused by a converging lens. Does a diverging lens cause spherical aberration?

Spherical aberrations occur because the focal points of rays far from the principal axis of a spherical lens (or mirror) are different from the focal points of rays of the same wavelength passing near the axis. Figure 36.31 illustrates spherical aberration for parallel rays passing through a converging lens. Rays passing through points near the center of the lens are imaged farther from the lens than rays passing through points near the edges.

Many cameras have an adjustable aperture to control light intensity and reduce spherical aberration. (An aperture is an opening that controls the amount of light passing through the lens.) Sharper images are produced as the aperture size is reduced because with a small aperture only the central portion of the lens is exposed to the light; as a result, a greater percentage of the rays are paraxial. At the



(a) (b) (c)

Lens aberrations. (a) *Spherical aberration* occurs when light passing through the lens at different distances from the principal axis is focused at different points. (b) *Astigmatism* occurs for objects not located on the principal axis of the lens. (c) *Coma* occurs as light passing through the lens far from the principal axis and light passing near the center of the lens focus at different parts of the focal plane.

same time, however, less light passes through the lens. To compensate for this lower light intensity, a longer exposure time is used.

In the case of mirrors used for very distant objects, spherical aberration can be minimized through the use of a parabolic reflecting surface rather than a spherical surface. Parabolic surfaces are not used often, however, because those with high-quality optics are very expensive to make. Parallel light rays incident on a parabolic surface focus at a common point, regardless of their distance from the principal axis. Parabolic reflecting surfaces are used in many astronomical telescopes to enhance image quality.

Chromatic Aberrations

The fact that different wavelengths of light refracted by a lens focus at different points gives rise to chromatic aberrations. In Chapter 35 we described how the index of refraction of a material varies with wavelength. For instance, when white light passes through a lens, violet rays are refracted more than red rays (Fig. 36.32). From this we see that the focal length is greater for red light than for violet light. Other wavelengths (not shown in Fig. 36.32) have focal points intermediate between those of red and violet.

Chromatic aberration for a diverging lens also results in a shorter focal length for violet light than for red light, but on the front side of the lens. Chromatic aberration can be greatly reduced by combining a converging lens made of one type of glass and a diverging lens made of another type of glass.

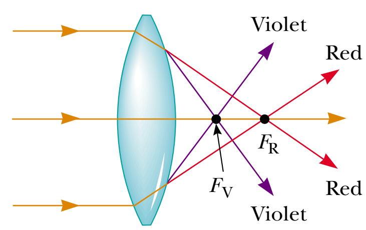


Figure 36.32 Chromatic aberration caused by a converging lens. Rays of different wavelengths focus at different points.

Optional Section

36.6 THE CAMERA

The photographic **camera** is a simple optical instrument whose essential features are shown in Figure 36.33. It consists of a light-tight box, a converging lens that produces a real image, and a film behind the lens to receive the image. One focuses the camera by varying the distance between lens and film. This is accom-

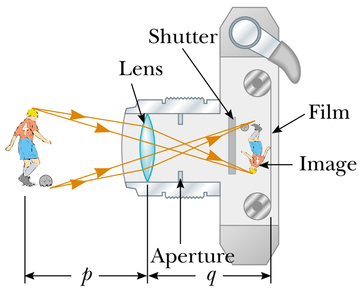


Figure 36.33 Cross-sectional view of a simple camera. Note that in reality, $p \gg q$.

plished with an adjustable bellows in older-style cameras and with some other mechanical arrangement in modern cameras. For proper focusing—which is necessary for the formation of sharp images—the lens-to-film distance depends on the object distance as well as on the focal length of the lens.

The shutter, positioned behind the lens, is a mechanical device that is opened for selected time intervals, called *exposure times.* One can photograph moving objects by using short exposure times, or photograph dark scenes (with low light levels) by using long exposure times. If this adjustment were not available, it would be impossible to take stop-action photographs. For example, a rapidly moving vehicle could move enough in the time that the shutter was open to produce a blurred image. Another major cause of blurred images is the movement of the camera while the shutter is open. To prevent such movement, either short exposure times or a tripod should be used, even for stationary objects. Typical shutter speeds (that is, exposure times) are 1/30, 1/60, 1/125, and 1/250 s. For handheld cameras, the use of slower speeds can result in blurred images (due to movement), but the use of faster speeds reduces the gathered light intensity. In practice, stationary objects are normally shot with an intermediate shutter speed of 1/60 s.

More expensive cameras have an aperture of adjustable diameter to further control the intensity of the light reaching the film. As noted earlier, when an aperture of small diameter is used, only light from the central portion of the lens reaches the film; in this way spherical aberration is reduced.

The intensity I of the light reaching the film is proportional to the area of the lens. Because this area is proportional to the square of the diameter D, we conclude that I is also proportional to D^2. Light intensity is a measure of the rate at which energy is received by the film per unit area of the image. Because the area of the image is proportional to q^2, and $q \approx f$ (when $p \gg f$, so p can be approximated as infinite), we conclude that the intensity is also proportional to $1/f^2$, and thus $I \propto D^2/f^2$. The brightness of the image formed on the film depends on the light intensity, so we see that the image brightness depends on both the focal length and the diameter of the lens.

The ratio f/D is called the ***f*-number** of a lens:

$$f\text{-number} \equiv \frac{f}{D} \tag{36.14}$$

Hence, the intensity of light incident on the film can be expressed as

$$I \propto \frac{1}{(f/D)^2} \propto \frac{1}{(f\text{-number})^2} \tag{36.15}$$

The *f*-number is often given as a description of the lens "speed." The lower the *f*-number, the wider the aperture and the higher the rate at which energy from the light exposes the film—thus, a lens with a low *f*-number is a "fast" lens. The conventional notation for an *f*-number is "$f/$" followed by the actual number. For example, "$f/4$" means an *f*-number of 4—it *does not* mean to divide f by 4! Extremely fast lenses, which have *f*-numbers as low as approximately $f/1.2$, are expensive because it is very difficult to keep aberrations acceptably small with light rays passing through a large area of the lens. Camera lens systems (that is, combinations of lenses with adjustable apertures) are often marked with multiple *f*-numbers, usually $f/2.8$, $f/4$, $f/5.6$, $f/8$, $f/11$, and $f/16$. Any one of these settings can be selected by adjusting the aperture, which changes the value of D. Increasing the setting from one *f*-number to the next higher value (for example, from $f/2.8$ to $f/4$) decreases the area of the aperture by a factor of two. The lowest *f*-number set-

ting on a camera lens corresponds to a wide-open aperture and the use of the maximum possible lens area.

Simple cameras usually have a fixed focal length and a fixed aperture size, with an f-number of about $f/11$. This high value for the f-number allows for a large **depth of field,** meaning that objects at a wide range of distances from the lens form reasonably sharp images on the film. In other words, the camera does not have to be focused.

EXAMPLE 36.14 Finding the Correct Exposure Time

The lens of a certain 35-mm camera (where 35 mm is the width of the film strip) has a focal length of 55 mm and a speed (an f-number) of $f/1.8$. The correct exposure time for this speed under certain conditions is known to be (1/500) s. (a) Determine the diameter of the lens.

Solution From Equation 36.14, we find that

$$D = \frac{f}{f\text{-number}} = \frac{55 \text{ mm}}{1.8} = \boxed{31 \text{ mm}}$$

(b) Calculate the correct exposure time if the f-number is changed to $f/4$ under the same lighting conditions.

Solution The total light energy hitting the film is proportional to the product of the intensity and the exposure time. If I is the light intensity reaching the film, then in a time t the energy per unit area received by the film is proportional to It. Comparing the two situations, we require that $I_1 t_1 = I_2 t_2$, where t_1 is the correct exposure time for $f/1.8$ and t_2 is the correct exposure time for $f/4$. Using this result together with Equation 36.15, we find that

$$\frac{t_1}{(f_1\text{-number})^2} = \frac{t_2}{(f_2\text{-number})^2}$$

$$t_2 = \left(\frac{f_2\text{-number}}{f_1\text{-number}}\right)^2 t_1$$

$$= \left(\frac{4}{1.8}\right)^2\left(\frac{1}{500} \text{ s}\right) \approx \boxed{\frac{1}{100} \text{ s}}$$

As the aperture size is reduced, exposure time must increase.

Optional Section

36.7 THE EYE

Like a camera, a normal eye focuses light and produces a sharp image. However, the mechanisms by which the eye controls the amount of light admitted and adjusts to produce correctly focused images are far more complex, intricate, and effective than those in even the most sophisticated camera. In all respects, the eye is a physiological wonder.

Figure 36.34 shows the essential parts of the human eye. Light entering the eye passes through a transparent structure called the *cornea,* behind which are a clear liquid (the *aqueous humor*), a variable aperture (the *pupil,* which is an opening in the *iris*), and the *crystalline lens.* Most of the refraction occurs at the outer surface of the eye, where the cornea is covered with a film of tears. Relatively little refraction occurs in the crystalline lens because the aqueous humor in contact with the lens has an average index of refraction close to that of the lens. The iris, which is the colored portion of the eye, is a muscular diaphragm that controls pupil size. The iris regulates the amount of light entering the eye by dilating the pupil in low-light conditions and contracting the pupil in high-light conditions. The f-number range of the eye is from about $f/2.8$ to $f/16$.

Close-up photograph of the cornea of the human eye.

The cornea–lens system focuses light onto the back surface of the eye, the *retina,* which consists of millions of sensitive receptors called *rods* and *cones.* When stimulated by light, these receptors send impulses via the optic nerve to the brain,

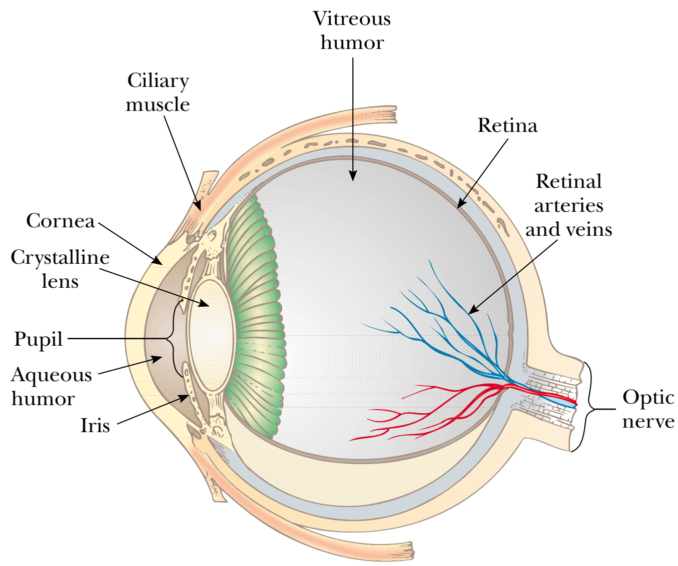


Figure 36.34 Essential parts of the eye.

where an image is perceived. By this process, a distinct image of an object is observed when the image falls on the retina.

The eye focuses on an object by varying the shape of the pliable crystalline lens through an amazing process called **accommodation.** An important component of accommodation is the *ciliary muscle,* which is situated in a circle around the rim of the lens. Thin filaments, called *zonules,* run from this muscle to the edge of the lens. When the eye is focused on a distant object, the ciliary muscle is relaxed, tightening the zonules that attach the muscle to the edge of the lens. The force of the zonules causes the lens to flatten, increasing its focal length. For an object distance of infinity, the focal length of the eye is equal to the fixed distance between lens and retina, about 1.7 cm. The eye focuses on nearby objects by tensing the ciliary muscle, which relaxes the zonules. This action allows the lens to bulge a bit, and its focal length decreases, resulting in the image being focused on the retina. All these lens adjustments take place so swiftly that we are not even aware of the change. In this respect, even the finest electronic camera is a toy compared with the eye.

QuickLab

Move this book toward your face until the letters just begin to blur. The distance from the book to your eyes is your near point.

Accommodation is limited in that objects that are very close to the eye produce blurred images. The **near point** is the closest distance for which the lens can accommodate to focus light on the retina. This distance usually increases with age and has an average value of 25 cm. Typically, at age 10 the near point of the eye is about 18 cm. It increases to about 25 cm at age 20, to 50 cm at age 40, and to 500 cm or greater at age 60. The **far point** of the eye represents the greatest distance for which the lens of the relaxed eye can focus light on the retina. A person with normal vision can see very distant objects, such as the Moon, and thus has a far point near infinity.

Recall that the light leaving the mirror in Figure 36.8 becomes white where it comes together but then diverges into separate colors again. Because nothing but air exists at the point where the rays cross (and hence nothing exists to cause the colors to separate again), seeing white light as a result of a combination of colors must be a visual illusion. In fact, this is the case. Only three types of color-sensitive

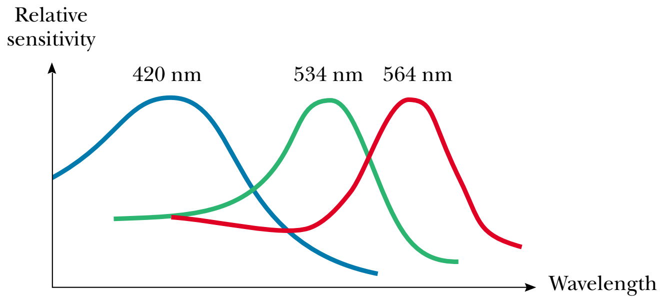


Figure 36.35 Approximate color sensitivity of the three types of cones in the retina.

cells are present in the retina; they are called red, green, and blue cones because of the peaks of the color ranges to which they respond (Fig. 36.35). If the red and green cones are stimulated simultaneously (as would be the case if yellow light were shining on them), the brain interprets what we see as yellow. If all three types of cones are stimulated by the separate colors red, blue, and green, as in Figure 36.8, we see white. If all three types of cones are stimulated by light that contains *all* colors, such as sunlight, we again see white light.

Color televisions take advantage of this visual illusion by having only red, green, and blue dots on the screen. With specific combinations of brightness in these three primary colors, our eyes can be made to see any color in the rainbow. Thus, the yellow lemon you see in a television commercial is not really yellow, it is red and green! The paper on which this page is printed is made of tiny, matted, translucent fibers that scatter light in all directions; the resultant mixture of colors appears white to the eye. Snow, clouds, and white hair are not really white. In fact, there is no such thing as a white pigment. The appearance of these things is a consequence of the scattering of light containing all colors, which we interpret as white.

QuickLab

Pour a pile of salt or sugar into your palm. Compare its white appearance with the transparency of a single grain.

Conditions of the Eye

When the eye suffers a mismatch between the focusing range of the lens–cornea system and the length of the eye, with the result that light rays reach the retina before they converge to form an image, as shown in Figure 36.36a, the condition is known as **farsightedness** (or *hyperopia*). A farsighted person can usually see faraway objects clearly but not nearby objects. Although the near point of a normal eye is approximately 25 cm, the near point of a farsighted person is much farther away. The eye of a farsighted person tries to focus by accommodation—that is, by shortening its focal length. This works for distant objects, but because the focal length of the farsighted eye is greater than normal, the light from nearby objects cannot be brought to a sharp focus before it reaches the retina, and it thus causes a blurred image. The refracting power in the cornea and lens is insufficient to focus the light from all but distant objects satisfactorily. The condition can be corrected by placing a converging lens in front of the eye, as shown in Figure 36.36b. The lens refracts the incoming rays more toward the principal axis before entering the eye, allowing them to converge and focus on the retina.

A person with **nearsightedness** (or *myopia*), another mismatch condition, can focus on nearby objects but not on faraway objects. In the case of *axial myopia,* the nearsightedness is caused by the lens being too far from the retina. In *refractive my-*

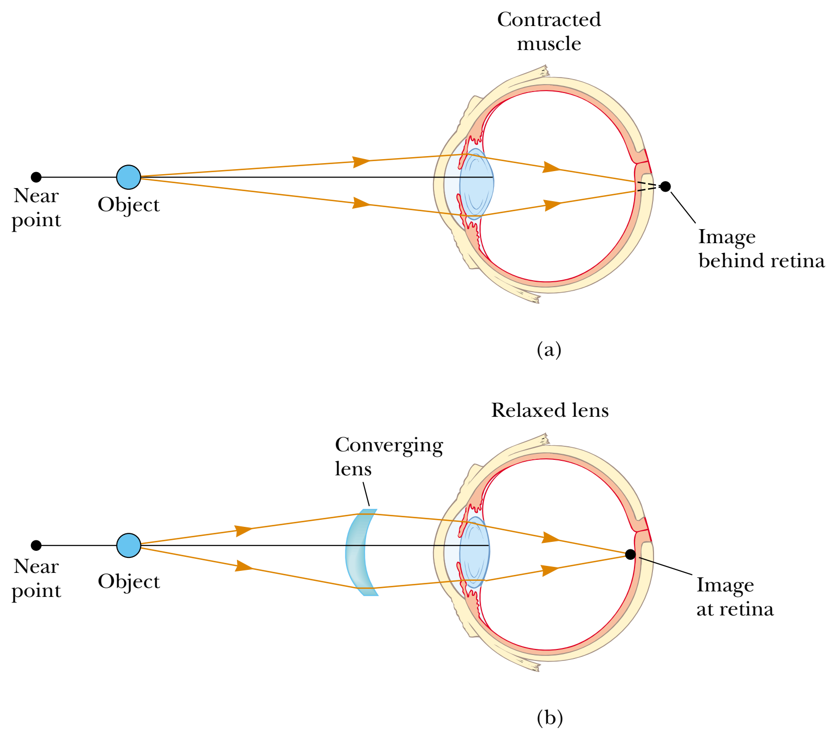


Figure 36.36 (a) When a farsighted eye looks at an object located between the near point and the eye, the image point is behind the retina, resulting in blurred vision. The eye muscle contracts to try to bring the object into focus. (b) Farsightedness is corrected with a converging lens.

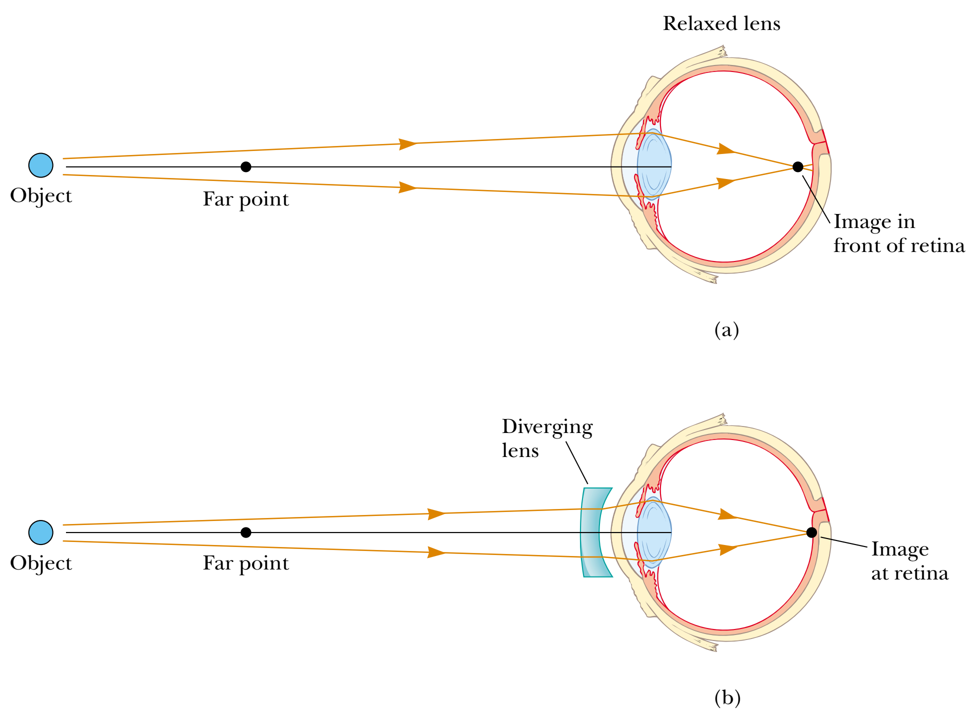


Figure 36.37 (a) When a nearsighted eye looks at an object that lies beyond the eye's far point, the image is formed in front of the retina, resulting in blurred vision. (b) Nearsightedness is corrected with a diverging lens.

opia, the lens–cornea system is too powerful for the length of the eye. The far point of the nearsighted eye is not infinity and may be less than 1 m. The maximum focal length of the nearsighted eye is insufficient to produce a sharp image on the retina, and rays from a distant object converge to a focus in front of the retina. They then continue past that point, diverging before they finally reach the retina and causing blurred vision (Fig. 36.37a). Nearsightedness can be corrected with a diverging lens, as shown in Figure 36.37b. The lens refracts the rays away from the principal axis before they enter the eye, allowing them to focus on the retina.

Quick Quiz 36.5

Which glasses in Figure 36.38 correct nearsightedness and which correct farsightedness?

(a) (b)

Figure 36.38

Beginning in middle age, most people lose some of their accommodation ability as the ciliary muscle weakens and the lens hardens. Unlike farsightedness, which is a mismatch between focusing power and eye length, **presbyopia** (literally, "old-age vision") is due to a reduction in accommodation ability. The cornea and lens do not have sufficient focusing power to bring nearby objects into focus on the retina. The symptoms are the same as those of farsightedness, and the condition can be corrected with converging lenses.

In the eye defect known as **astigmatism,** light from a point source produces a line image on the retina. This condition arises when either the cornea or the lens or both are not perfectly symmetric. Astigmatism can be corrected with lenses that have different curvatures in two mutually perpendicular directions.

Optometrists and ophthalmologists usually prescribe lenses[1] measured in **diopters:**

The **power** P of a lens in diopters equals the inverse of the focal length in meters: $P = 1/f$.

For example, a converging lens of focal length $+20$ cm has a power of $+5.0$ diopters, and a diverging lens of focal length -40 cm has a power of -2.5 diopters.

EXAMPLE 36.15 A Case of Nearsightedness

A particular nearsighted person is unable to see objects clearly when they are beyond 2.5 m away (the far point of this particular eye). What should the focal length be in a lens prescribed to correct this problem?

Solution The purpose of the lens in this instance is to "move" an object from infinity to a distance where it can be seen clearly. This is accomplished by having the lens produce an image at the far point. From the thin-lens equation, we have

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\infty} + \frac{1}{-2.5\text{ m}} = \frac{1}{f}$$

$$f = \boxed{-2.5\text{ m}}$$

Why did we use a negative sign for the image distance? As you should have suspected, the lens must be a diverging lens (one with a negative focal length) to correct nearsightedness.

Exercise What is the power of this lens?

Answer -0.40 diopter.

Optional Section

36.8 THE SIMPLE MAGNIFIER

The simple magnifier consists of a single converging lens. As the name implies, this device increases the apparent size of an object.

Suppose an object is viewed at some distance p from the eye, as illustrated in Figure 36.39. The size of the image formed at the retina depends on the angle θ subtended by the object at the eye. As the object moves closer to the eye, θ increases and a larger image is observed. However, an average normal eye cannot focus on an object closer than about 25 cm, the near point (Fig. 36.40a). Therefore, θ is maximum at the near point.

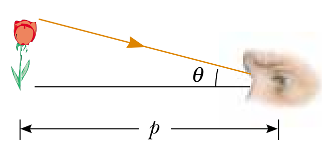


Figure 36.39 The size of the image formed on the retina depends on the angle θ subtended at the eye.

To further increase the apparent angular size of an object, a converging lens can be placed in front of the eye as in Figure 36.40b, with the object located at point O, just inside the focal point of the lens. At this location, the lens forms a virtual, upright, enlarged image. We define **angular magnification** m as the ratio of the angle subtended by an object with a lens in use (angle θ in Fig. 36.40b) to the angle subtended by the object placed at the near point with no lens in use (angle

[1] The word *lens* comes from *lentil*, the name of an Italian legume. (You may have eaten lentil soup.) Early eyeglasses were called "glass lentils" because the biconvex shape of their lenses resembled the shape of a lentil. The first lenses for farsightedness and presbyopia appeared around 1280; concave eyeglasses for correcting nearsightedness did not appear for more than 100 years after that.

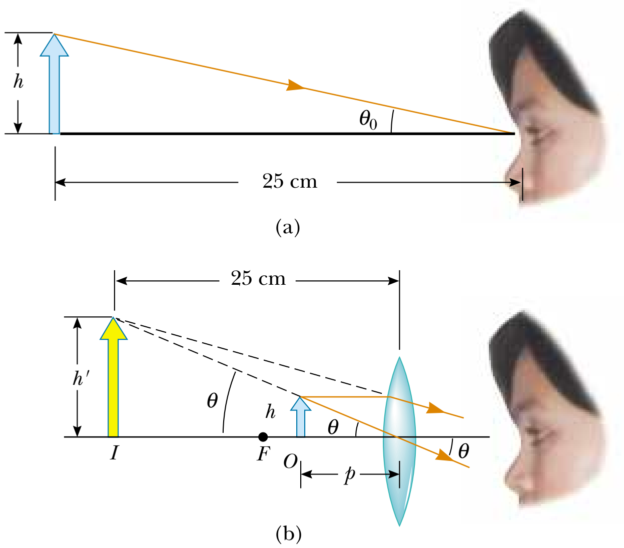


Figure 36.40 (a) An object placed at the near point of the eye ($p = 25$ cm) subtends an angle $\theta_0 \approx h/25$ at the eye. (b) An object placed near the focal point of a converging lens produces a magnified image that subtends an angle $\theta \approx h'/25$ at the eye.

θ_0 in Fig. 36.40a):

$$m \equiv \frac{\theta}{\theta_0} \tag{36.16}$$

Angular magnification with the object at the near point

The angular magnification is a maximum when the image is at the near point of the eye—that is, when $q = -25$ cm. The object distance corresponding to this image distance can be calculated from the thin-lens equation:

$$\frac{1}{p} + \frac{1}{-25\text{ cm}} = \frac{1}{f}$$

$$p = \frac{25f}{25 + f}$$

where f is the focal length of the magnifier in centimeters. If we make the small-angle approximations

$$\tan\theta_0 \approx \theta_0 \approx \frac{h}{25} \qquad \text{and} \qquad \tan\theta \approx \theta \approx \frac{h}{p} \tag{36.17}$$

Equation 36.16 becomes

$$m_{\text{max}} = \frac{\theta}{\theta_0} = \frac{h/p}{h/25} = \frac{25}{p} = \frac{25}{25f/(25+f)}$$

$$m_{\text{max}} = 1 + \frac{25\text{ cm}}{f} \tag{36.18}$$

Although the eye can focus on an image formed anywhere between the near point and infinity, it is most relaxed when the image is at infinity. For the image formed by the magnifying lens to appear at infinity, the object has to be at the focal point of the lens. In this case, Equations 36.17 become

$$\theta_0 \approx \frac{h}{25} \qquad \text{and} \qquad \theta \approx \frac{h}{f}$$

and the magnification is

$$m_{\min} = \frac{\theta}{\theta_0} = \frac{25 \text{ cm}}{f} \qquad \textbf{(36.19)}$$

With a single lens, it is possible to obtain angular magnifications up to about 4 without serious aberrations. Magnifications up to about 20 can be achieved by using one or two additional lenses to correct for aberrations.

EXAMPLE 36.16 Maximum Magnification of a Lens

What is the maximum magnification that is possible with a lens having a focal length of 10 cm, and what is the magnification of this lens when the eye is relaxed?

Solution The maximum magnification occurs when the image is located at the near point of the eye. Under these circumstances, Equation 36.18 gives

$$m_{\max} = 1 + \frac{25 \text{ cm}}{f} = 1 + \frac{25 \text{ cm}}{10 \text{ cm}} = \boxed{3.5}$$

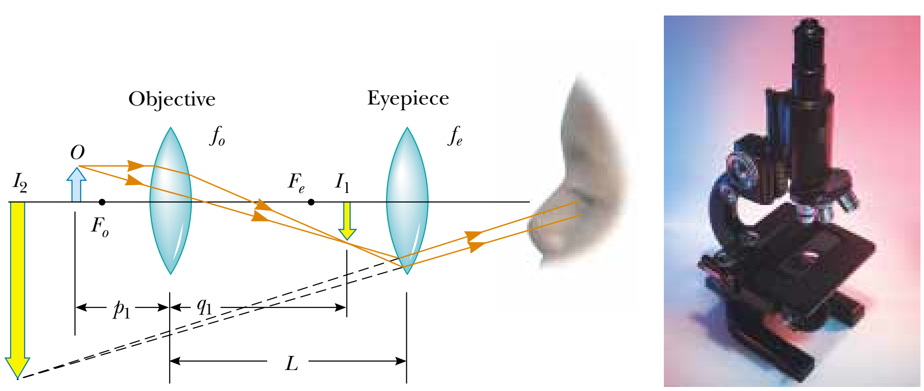
When the eye is relaxed, the image is at infinity. In this case, we use Equation 36.19:

$$m_{\min} = \frac{25 \text{ cm}}{f} = \frac{25 \text{ cm}}{10 \text{ cm}} = \boxed{2.5}$$

Optional Section

36.9 THE COMPOUND MICROSCOPE

A simple magnifier provides only limited assistance in inspecting minute details of an object. Greater magnification can be achieved by combining two lenses in a device called a **compound microscope,** a schematic diagram of which is shown in Figure 36.41a. It consists of one lens, the *objective,* that has a very short focal length



(a) (b)

Figure 36.41 (a) Diagram of a compound microscope, which consists of an objective lens and an eyepiece lens. (b) A compound microscope. The three-objective turret allows the user to choose from several powers of magnification. Combinations of eyepieces with different focal lengths and different objectives can produce a wide range of magnifications.

$f_o < 1$ cm and a second lens, the *eyepiece,* that has a focal length f_e of a few centimeters. The two lenses are separated by a distance L that is much greater than either f_o or f_e. The object, which is placed just outside the focal point of the objective, forms a real, inverted image at I_1, and this image is located at or close to the focal point of the eyepiece. The eyepiece, which serves as a simple magnifier, produces at I_2 a virtual, inverted image of I_1. The lateral magnification M_1 of the first image is $-q_1/p_1$. Note from Figure 36.41a that q_1 is approximately equal to L and that the object is very close to the focal point of the objective: $p_1 \approx f_o$. Thus, the lateral magnification by the objective is

$$M_1 \approx -\frac{L}{f_o}$$

The angular magnification by the eyepiece for an object (corresponding to the image at I_1) placed at the focal point of the eyepiece is, from Equation 36.19,

$$m_e = \frac{25 \text{ cm}}{f_e}$$

The overall magnification of the compound microscope is defined as the product of the lateral and angular magnifications:

$$M = M_1 m_e = -\frac{L}{f_o}\left(\frac{25 \text{ cm}}{f_e}\right) \tag{36.20}$$

The negative sign indicates that the image is inverted.

The microscope has extended human vision to the point where we can view previously unknown details of incredibly small objects. The capabilities of this instrument have steadily increased with improved techniques for precision grinding of lenses. An often-asked question about microscopes is: "If one were extremely patient and careful, would it be possible to construct a microscope that would enable the human eye to see an atom?" The answer is no, as long as light is used to illuminate the object. The reason is that, for an object under an optical microscope (one that uses visible light) to be seen, the object must be at least as large as a wavelength of light. Because the diameter of any atom is many times smaller than the wavelengths of visible light, the mysteries of the atom must be probed using other types of "microscopes."

The ability to use other types of waves to "see" objects also depends on wavelength. We can illustrate this with water waves in a bathtub. Suppose you vibrate your hand in the water until waves having a wavelength of about 15 cm are moving along the surface. If you hold a small object, such as a toothpick, so that it lies in the path of the waves, it does not appreciably disturb the waves; they continue along their path "oblivious" to it. Now suppose you hold a larger object, such as a toy sailboat, in the path of the 15-cm waves. In this case, the waves are considerably disturbed by the object. Because the toothpick was smaller than the wavelength of the waves, the waves did not "see" it (the intensity of the scattered waves was low). Because it is about the same size as the wavelength of the waves, however, the boat creates a disturbance. In other words, the object acts as the source of scattered waves that appear to come from it.

Light waves behave in this same general way. The ability of an optical microscope to view an object depends on the size of the object relative to the wavelength of the light used to observe it. Hence, we can never observe atoms with an optical

microscope[2] because their dimensions are small (≈ 0.1 nm) relative to the wavelength of the light (≈ 500 nm).

Optional Section

36.10 THE TELESCOPE

14.1 & 14.9

Two fundamentally different types of **telescopes** exist; both are designed to aid in viewing distant objects, such as the planets in our Solar System. The **refracting telescope** uses a combination of lenses to form an image, and the **reflecting telescope** uses a curved mirror and a lens.

The lens combination shown in Figure 36.42a is that of a refracting telescope. Like the compound microscope, this telescope has an objective and an eyepiece. The two lenses are arranged so that the objective forms a real, inverted image of the distant object very near the focal point of the eyepiece. Because the object is essentially at infinity, this point at which I_1 forms is the focal point of the objective. Hence, the two lenses are separated by a distance $f_o + f_e$, which corresponds to the length of the telescope tube. The eyepiece then forms, at I_2, an enlarged, inverted image of the image at I_1.

The angular magnification of the telescope is given by θ/θ_o, where θ_o is the angle subtended by the object at the objective and θ is the angle subtended by the final image at the viewer's eye. Consider Figure 36.42a, in which the object is a very great distance to the left of the figure. The angle θ_o (to the *left* of the objective) subtended by the object at the objective is the same as the angle (to the *right* of the objective) subtended by the first image at the objective. Thus,

$$\tan \theta_o \approx \theta_o \approx -\frac{h'}{f_o}$$

where the negative sign indicates that the image is inverted.

The angle θ subtended by the final image at the eye is the same as the angle that a ray coming from the tip of I_1 and traveling parallel to the principal axis makes with the principal axis after it passes through the lens. Thus,

$$\tan \theta \approx \theta \approx \frac{h'}{f_e}$$

We have not used a negative sign in this equation because the final image is not inverted; the object creating this final image I_2 is I_1, and both it and I_2 point in the same direction. To see why the adjacent side of the triangle containing angle θ is f_e and not $2f_e$, note that we must use only the bent length of the refracted ray. Hence, the angular magnification of the telescope can be expressed as

$$m = \frac{\theta}{\theta_o} = \frac{h'/f_e}{-h'/f_o} = -\frac{f_o}{f_e} \qquad \textbf{(36.21)}$$

and we see that the angular magnification of a telescope equals the ratio of the objective focal length to the eyepiece focal length. The negative sign indicates that the image is inverted.

Quick Quiz 36.6

Why isn't the lateral magnification given by Equation 36.1 a useful concept for telescopes?

[2] Single-molecule near-field optic studies are routinely performed with visible light having wavelengths of about 500 nm. The technique uses very small apertures to produce images having resolution as small as 10 nm.

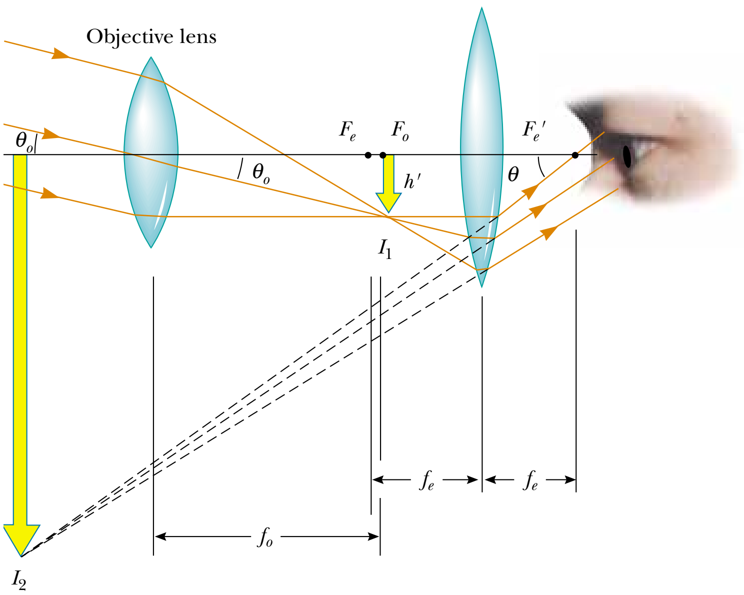


Figure 36.42 (a) Lens arrangement in a refracting telescope, with the object at infinity. (b) A refracting telescope.

When we look through a telescope at such relatively nearby objects as the Moon and the planets, magnification is important. However, stars are so far away that they always appear as small points of light no matter how great the magnification. A large research telescope that is used to study very distant objects must have a great diameter to gather as much light as possible. It is difficult and expensive to manufacture large lenses for refracting telescopes. Another difficulty with large lenses is that their weight leads to sagging, which is an additional source of aberration. These problems can be partially overcome by replacing the objective with a concave mirror, which results in a reflecting telescope. Because light is reflected from the mirror and does not pass through a lens, the mirror can have rigid supports on the back side. Such supports eliminate the problem of sagging.

Figure 36.43 shows the design for a typical reflecting telescope. Incoming light rays pass down the barrel of the telescope and are reflected by a parabolic mirror at the base. These rays converge toward point *A* in the figure, where an image would be formed. However, before this image is formed, a small, flat mirror M reflects the light toward an opening in the side of the tube that passes into an eyepiece. This particular design is said to have a Newtonian focus because Newton developed it. Note that in the reflecting telescope the light never passes through glass (except through the small eyepiece). As a result, problems associated with chromatic aberration are virtually eliminated.

The largest reflecting telescopes in the world are at the Keck Observatory on Mauna Kea, Hawaii. The site includes two telescopes with diameters of 10 m, each containing 36 hexagonally shaped, computer-controlled mirrors that work together to form a large reflecting surface. In contrast, the largest refracting telescope in the world, at the Yerkes Observatory in Williams Bay, Wisconsin, has a diameter of only 1 m.

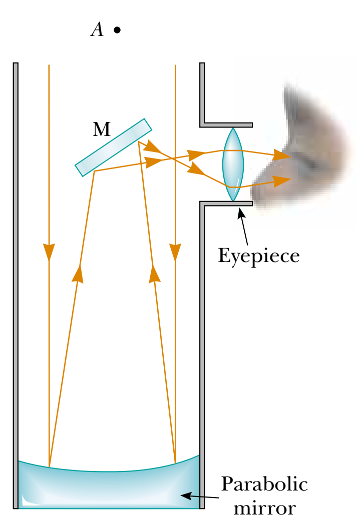


Figure 36.43 A Newtonian-focus reflecting telescope.

web

For more information on the Keck telescopes, visit **http://www2.keck.hawaii.edu:3636/**

SUMMARY

The **lateral magnification** M of a mirror or lens is defined as the ratio of the image height h' to the object height h:

$$M = \frac{h'}{h} \tag{36.1}$$

In the paraxial ray approximation, the object distance p and image distance q for a spherical mirror of radius R are related by the **mirror equation:**

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{R} = \frac{1}{f} \tag{36.4, 36.6}$$

where $f = R/2$ is the **focal length** of the mirror.

An image can be formed by refraction from a spherical surface of radius R. The object and image distances for refraction from such a surface are related by

$$\frac{n_1}{p} + \frac{n_2}{q} = \frac{n_2 - n_1}{R} \tag{36.8}$$

where the light is incident in the medium for which the index of refraction is n_1 and is refracted in the medium for which the index of refraction is n_2.

The inverse of the **focal length** f of a thin lens surrounded by air is given by the **lens makers' equation:**

$$\frac{1}{f} = (n - 1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \tag{36.11}$$

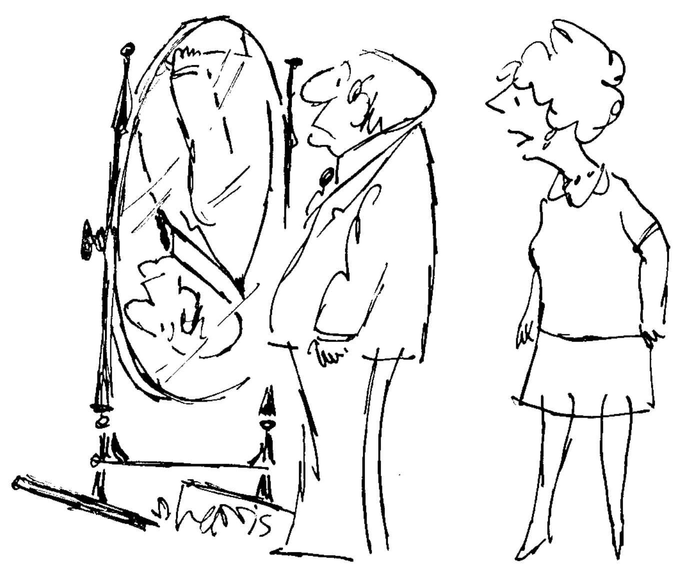
Converging lenses have positive focal lengths, and **diverging lenses** have negative focal lengths.

For a thin lens, and in the paraxial ray approximation, the object and image distances are related by the **thin-lens equation:**

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{f} \tag{36.12}$$

QUESTIONS

1. What is wrong with the caption of the cartoon shown in Figure Q36.1?
2. Using a simple ray diagram, such as the one shown in Figure 36.2, show that a flat mirror whose top is at eye level need not be as long as you are for you to see your entire body in it.
3. Consider a concave spherical mirror with a real object. Is the image always inverted? Is the image always real? Give conditions for your answers.
4. Repeat the preceding question for a convex spherical mirror.
5. Why does a clear stream of water, such as a creek, always appear to be shallower than it actually is? By how much is its depth apparently reduced?
6. Consider the image formed by a thin converging lens. Under what conditions is the image (a) inverted, (b) upright, (c) real, (d) virtual, (e) larger than the object, and (f) smaller than the object?
7. Repeat Question 6 for a thin diverging lens.
8. Use the lens makers' equation to verify the sign of the focal length of each of the lenses in Figure 36.26.



"Most mirrors reverse left and right. This one reverses top and bottom."

Figure Q36.1

9. If a cylinder of solid glass or clear plastic is placed above the words LEAD OXIDE and viewed from the side as shown in Figure Q36.9, the LEAD appears inverted but the OXIDE does not. Explain.

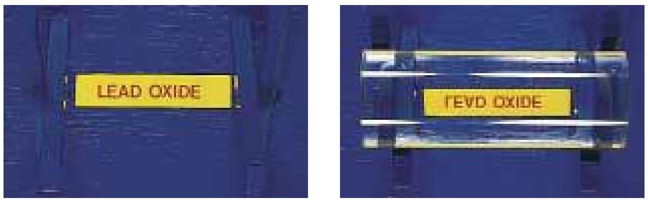


Figure Q36.9

10. If the camera "sees" a movie actor's reflection in a mirror, what does the actor see in the mirror?
11. Explain why a mirror cannot give rise to chromatic aberration.
12. Why do some automobile mirrors have printed on them the statement "Objects in mirror are closer than they appear"? (See Fig. Q36.12.)

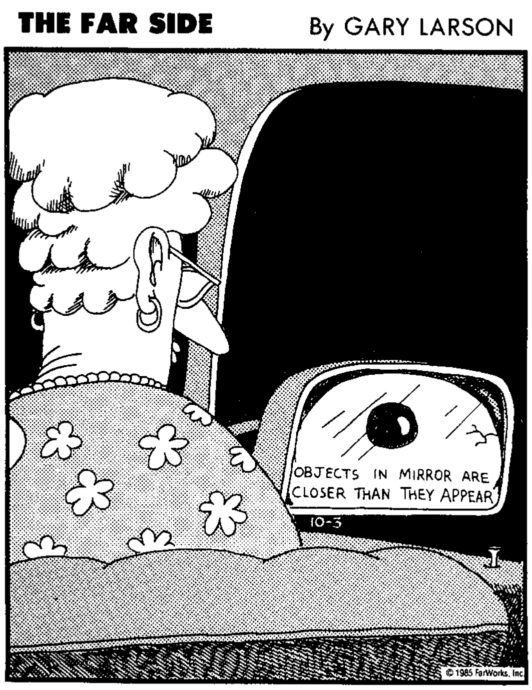


Figure Q36.12

13. Why do some emergency vehicles have the symbol ƎƆИA⅃UꓭMA written on the front?
14. Explain why a fish in a spherical goldfish bowl appears larger than it really is.
15. Lenses used in eyeglasses, whether converging or diverging, are always designed such that the middle of the lens curves away from the eye, like the center lenses of Figure 36.26a and b. Why?
16. A mirage is formed when the air gets gradually cooler with increasing altitude. What might happen if the air grew gradually warmer with altitude? This often happens over bodies of water or snow-covered ground; the effect is called *looming*.
17. Consider a spherical concave mirror, with an object positioned to the left of the mirror beyond the focal point. Using ray diagrams, show that the image moves to the left as the object approaches the focal point.
18. In a Jules Verne novel, a piece of ice is shaped into a magnifying lens to focus sunlight to start a fire. Is this possible?
19. The *f*-number of a camera is the focal length of the lens divided by its aperture (or diameter). How can the *f*-number of the lens be changed? How does changing this number affect the required exposure time?
20. A solar furnace can be constructed through the use of a concave mirror to reflect and focus sunlight into a furnace enclosure. What factors in the design of the reflecting mirror would guarantee very high temperatures?
21. One method for determining the position of an image, either real or virtual, is by means of *parallax*. If a finger or another object is placed at the position of the image, as shown in Figure Q36.21, and the finger and the image are viewed simultaneously (the image is viewed through the lens if it is virtual), the finger and image have the same parallax; that is, if the image is viewed from different positions, it will appear to move along with the finger. Use this method to locate the image formed by a lens. Explain why the method works.

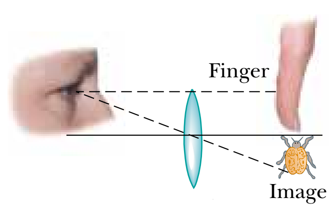


Figure Q36.21

22. Figure Q36.22 shows a lithograph by M. C. Escher titled *Hand with Reflection Sphere (Self-Portrait in Spherical Mirror)*. Escher had this to say about the work: "The picture shows a spherical mirror, resting on a left hand. But as a print is the reverse of the original drawing on stone, it was my right hand that you see depicted. (Being left-handed, I needed my left hand to make the drawing.) Such a globe reflection collects almost one's whole surroundings in one disk-shaped image. The whole room, four walls, the

floor, and the ceiling, everything, albeit distorted, is compressed into that one small circle. Your own head, or more exactly the point between your eyes, is the absolute center. No matter how you turn or twist yourself, you can't get out of that central point. You are immovably the focus, the unshakable core, of your world." Comment on the accuracy of Escher's description.

23. You can make a corner reflector by placing three flat mirrors in the corner of a room where the ceiling meets the walls. Show that no matter where you are in the room, you can see yourself reflected in the mirrors—upside down.

Figure Q36.22

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 36.1 Images Formed by Flat Mirrors

1. Does your bathroom mirror show you older or younger than you actually are? Compute an order-of-magnitude estimate for the age difference, based on data that you specify.

2. In a church choir loft, two parallel walls are 5.30 m apart. The singers stand against the north wall. The organist faces the south wall, sitting 0.800 m away from it. To enable her to see the choir, a flat mirror 0.600 m wide is mounted on the south wall, straight in front of her. What width of the north wall can she see? *Hint:* Draw a top-view diagram to justify your answer.

3. Determine the minimum height of a vertical flat mirror in which a person 5′10″ in height can see his or her full image. (A ray diagram would be helpful.)

4. Two flat mirrors have their reflecting surfaces facing each other, with an edge of one mirror in contact with an edge of the other, so that the angle between the mirrors is α. When an object is placed between the mirrors, a number of images are formed. In general, if the angle α is such that $n\alpha = 360°$, where n is an integer, the number of images formed is $n - 1$. Graphically, find all the image positions for the case $n = 6$ when a point object is between the mirrors (but not on the angle bisector).

5. A person walks into a room with two flat mirrors on opposite walls, which produce multiple images. When the person is 5.00 ft from the mirror on the left wall and 10.0 ft from the mirror on the right wall, find the distances from that person to the first three images seen in the mirror on the left.

Section 36.2 Images Formed by Spherical Mirrors

6. A concave spherical mirror has a radius of curvature of 20.0 cm. Find the location of the image for object distances of (a) 40.0 cm, (b) 20.0 cm, and (c) 10.0 cm. For each case, state whether the image is real or virtual and upright or inverted, and find the magnification.

7. At an intersection of hospital hallways, a convex mirror is mounted high on a wall to help people avoid collisions. The mirror has a radius of curvature of 0.550 m. Locate and describe the image of a patient 10.0 m from the mirror. Determine the magnification.

8. A large church has a niche in one wall. On the floor plan it appears as a semicircular indentation of radius 2.50 m. A worshiper stands on the center line of the niche, 2.00 m out from its deepest point, and whispers a prayer. Where is the sound concentrated after reflection from the back wall of the niche?

WEB 9. A spherical convex mirror has a radius of curvature of 40.0 cm. Determine the position of the virtual image and the magnification (a) for an object distance of 30.0 cm and (b) for an object distance of 60.0 cm. (c) Are the images upright or inverted?

10. The height of the real image formed by a concave mirror is four times the object height when the object is 30.0 cm in front of the mirror. (a) What is the radius of curvature of the mirror? (b) Use a ray diagram to locate this image.

11. A concave mirror has a radius of curvature of 60.0 cm. Calculate the image position and magnification of an object placed in front of the mirror (a) at a distance of 90.0 cm and (b) at a distance of 20.0 cm. (c) In each case, draw ray diagrams to obtain the image characteristics.

12. A concave mirror has a focal length of 40.0 cm. Determine the object position for which the resulting image is upright and four times the size of the object.

13. A spherical mirror is to be used to form, on a screen 5.00 m from the object, an image five times the size of the object. (a) Describe the type of mirror required. (b) Where should the mirror be positioned relative to the object?

14. A rectangle 10.0 cm $\times$ 20.0 cm is placed so that its right edge is 40.0 cm to the left of a concave spherical mirror, as in Figure P36.14. The radius of curvature of the mirror is 20.0 cm. (a) Draw the image formed by this mirror. (b) What is the area of the image?

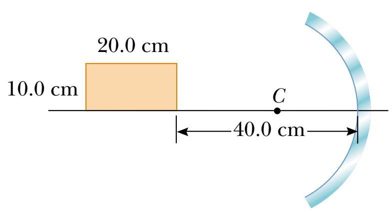


Figure P36.14

15. A dedicated sports-car enthusiast polishes the inside and outside surfaces of a hubcap that is a section of a sphere. When she looks into one side of the hubcap, she sees an image of her face 30.0 cm in back of the hubcap. She then flips the hubcap over and sees another image of her face 10.0 cm in back of the hubcap. (a) How far is her face from the hubcap? (b) What is the radius of curvature of the hubcap?

16. An object is 15.0 cm from the surface of a reflective spherical Christmas-tree ornament 6.00 cm in diameter. What are the magnification and position of the image?

17. A ball is dropped from rest 3.00 m directly above the vertex of a concave mirror that has a radius of 1.00 m and lies in a horizontal plane. (a) Describe the motion of the ball's image in the mirror. (b) At what time do the ball and its image coincide?

Section 36.3 Images Formed by Refraction

18. A flint-glass plate ($n = 1.66$) rests on the bottom of an aquarium tank. The plate is 8.00 cm thick (vertical dimension) and covered with water ($n = 1.33$) to a depth of 12.0 cm. Calculate the apparent thickness of the plate as viewed from above the water. (Assume nearly normal incidence.)

19. A cubical block of ice 50.0 cm on a side is placed on a level floor over a speck of dust. Find the location of the image of the speck if the index of refraction of ice is 1.309.

20. A simple model of the human eye ignores its lens entirely. Most of what the eye does to light happens at the transparent cornea. Assume that this outer surface has a 6.00-mm radius of curvature, and assume that the eyeball contains just one fluid with an index of refraction of 1.40. Prove that a very distant object will be imaged on the retina, 21.0 mm behind the cornea. Describe the image.

21. A glass sphere ($n = 1.50$) with a radius of 15.0 cm has a tiny air bubble 5.00 cm above its center. The sphere is viewed looking down along the extended radius containing the bubble. What is the apparent depth of the bubble below the surface of the sphere?

22. A transparent sphere of unknown composition is observed to form an image of the Sun on the surface of the sphere opposite the Sun. What is the refractive index of the sphere material?

23. One end of a long glass rod ($n = 1.50$) is formed into a convex surface of radius 6.00 cm. An object is positioned in air along the axis of the rod. Find the image positions corresponding to object distances of (a) 20.0 cm, (b) 10.0 cm, and (c) 3.00 cm from the end of the rod.

24. A goldfish is swimming at 2.00 cm/s toward the front wall of a rectangular aquarium. What is the apparent speed of the fish as measured by an observer looking in from outside the front wall of the tank? The index of refraction of water is 1.33.

25. A goldfish is swimming inside a spherical plastic bowl of water, with an index of refraction of 1.33. If the goldfish is 10.0 cm from the wall of the 15.0-cm-radius bowl, where does it appear to an observer outside the bowl?

Section 36.4 Thin Lenses

26. A contact lens is made of plastic with an index of refraction of 1.50. The lens has an outer radius of curvature of + 2.00 cm and an inner radius of curvature of + 2.50 cm. What is the focal length of the lens?

WEB 27. The left face of a biconvex lens has a radius of curvature of magnitude 12.0 cm, and the right face has a radius of curvature of magnitude 18.0 cm. The index of refraction of the glass is 1.44. (a) Calculate the focal length of the lens. (b) Calculate the focal length if the radii of curvature of the two faces are interchanged.

28. A converging lens has a focal length of 20.0 cm. Locate the image for object distances of (a) 40.0 cm, (b) 20.0 cm, and (c) 10.0 cm. For each case, state whether the image is real or virtual and upright or inverted. Find the magnification in each case.

29. A thin lens has a focal length of 25.0 cm. Locate and describe the image when the object is placed (a) 26.0 cm and (b) 24.0 cm in front of the lens.

30. An object positioned 32.0 cm in front of a lens forms an image on a screen 8.00 cm behind the lens. (a) Find the focal length of the lens. (b) Determine the magnification. (c) Is the lens converging or diverging?

WEB **31.** The nickel's image in Figure P36.31 has twice the diameter of the nickel and is 2.84 cm from the lens. Determine the focal length of the lens.

Figure P36.31

32. A magnifying glass is a converging lens of focal length 15.0 cm. At what distance from a postage stamp should you hold this lens to get a magnification of $+2.00$?

33. A transparent photographic slide is placed in front of a converging lens with a focal length of 2.44 cm. The lens forms an image of the slide 12.9 cm from the slide. How far is the lens from the slide if the image is (a) real? (b) virtual?

34. A person looks at a gem with a jeweler's loupe—a converging lens that has a focal length of 12.5 cm. The loupe forms a virtual image 30.0 cm from the lens. (a) Determine the magnification. Is the image upright or inverted? (b) Construct a ray diagram for this arrangement.

35. Suppose an object has thickness dp so that it extends from object distance p to $p + dp$. Prove that the thickness dq of its image is given by $(-q^2/p^2)\,dp$, so the longitudinal magnification $dq/dp = -M^2$, where M is the lateral magnification.

36. The projection lens in a certain slide projector is a single thin lens. A slide 24.0 mm high is to be projected so that its image fills a screen 1.80 m high. The slide-to-screen distance is 3.00 m. (a) Determine the focal length of the projection lens. (b) How far from the slide should the lens of the projector be placed to form the image on the screen?

37. An object is positioned 20.0 cm to the left of a diverging lens with focal length $f = -32.0$ cm. Determine (a) the location and (b) the magnification of the image. (c) Construct a ray diagram for this arrangement.

38. Figure P36.38 shows a thin glass ($n = 1.50$) converging lens for which the radii of curvature are $R_1 = 15.0$ cm and $R_2 = -12.0$ cm. To the left of the lens is a cube with a face area of 100 cm^2. The base of the cube is on the axis of the lens, and the right face is 20.0 cm to the left of the lens. (a) Determine the focal length of the lens. (b) Draw the image of the square face formed by the lens. What type of geometric figure is this? (c) Determine the area of the image.

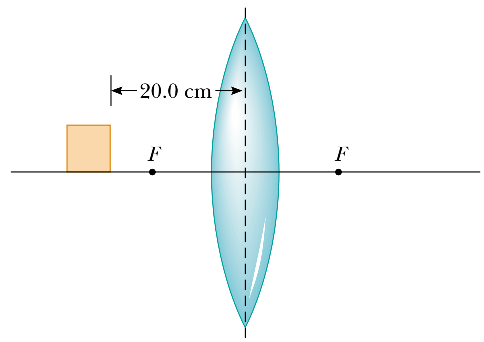


Figure P36.38

39. An object is 5.00 m to the left of a flat screen. A converging lens for which the focal length is $f = 0.800$ m is placed between object and screen. (a) Show that two lens positions exist that form images on the screen, and determine how far these positions are from the object. (b) How do the two images differ from each other?

40. An object is at a distance d to the left of a flat screen. A converging lens with focal length $f < d/4$ is placed between object and screen. (a) Show that two lens positions exist that form an image on the screen, and determine how far these positions are from the object. (b) How do the two images differ from each other?

41. Figure 36.33 diagrams a cross-section of a camera. It has a single lens with a focal length of 65.0 mm, which is to form an image on the film at the back of the camera. Suppose the position of the lens has been adjusted to focus the image of a distant object. How far and in what direction must the lens be moved to form a sharp image of an object that is 2.00 m away?

(Optional)
Section 36.5 Lens Aberrations

42. The magnitudes of the radii of curvature are 32.5 cm and 42.5 cm for the two faces of a biconcave lens. The glass has index 1.53 for violet light and 1.51 for red light. For a very distant object, locate and describe (a) the image formed by violet light and (b) the image formed by red light.

43. Two rays traveling parallel to the principal axis strike a large plano–convex lens having a refractive index of 1.60 (Fig. P36.43). If the convex face is spherical, a ray near the edge does not pass through the focal point (spherical aberration occurs). If this face has a radius of curvature of magnitude 20.0 cm and the two rays are $h_1 = 0.500$ cm and $h_2 = 12.0$ cm from the principal axis, find the difference in the positions where they cross the principal axis.

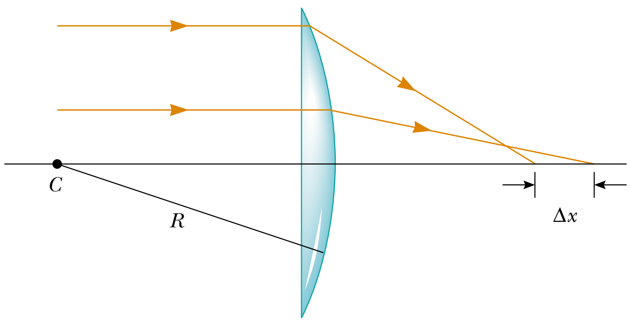


Figure P36.43

(Optional)
Section 36.7 The Eye

44. The accommodation limits for Nearsighted Nick's eyes are 18.0 cm and 80.0 cm. When he wears his glasses, he can see faraway objects clearly. At what minimum distance can he see objects clearly?

45. A nearsighted person cannot see objects clearly beyond 25.0 cm (her far point). If she has no astigmatism and contact lenses are prescribed for her, what power and type of lens are required to correct her vision?

46. A person sees clearly when he wears eyeglasses that have a power of -4.00 diopters and sit 2.00 cm in front of his eyes. If he wants to switch to contact lenses, which are placed directly on the eyes, what lens power should be prescribed?

(Optional)
Section 36.8 The Simple Magnifier
Section 36.9 The Compound Microscope
Section 36.10 The Telescope

47. A philatelist examines the printing detail on a stamp, using a biconvex lens with a focal length of 10.0 cm as a simple magnifier. The lens is held close to the eye, and the lens-to-object distance is adjusted so that the virtual image is formed at the normal near point (25.0 cm). Calculate the magnification.

48. A lens that has a focal length of 5.00 cm is used as a magnifying glass. (a) Where should the object be placed to obtain maximum magnification? (b) What is the magnification?

49. The distance between the eyepiece and the objective lens in a certain compound microscope is 23.0 cm. The focal length of the eyepiece is 2.50 cm, and that of the objective is 0.400 cm. What is the overall magnification of the microscope?

50. The desired overall magnification of a compound microscope is 140×. The objective alone produces a lateral magnification of 12.0×. Determine the required focal length of the eyepiece.

51. The Yerkes refracting telescope has a 1.00-m-diameter objective lens with a focal length of 20.0 m. Assume that it is used with an eyepiece that has a focal length of 2.50 cm. (a) Determine the magnification of the planet Mars as seen through this telescope. (b) Are the Martian polar caps seen right side up or upside down?

52. Astronomers often take photographs with the objective lens or the mirror of a telescope alone, without an eyepiece. (a) Show that the image size h' for this telescope is given by $h' = fh/(f - p)$, where h is the object size, f is the objective focal length, and p is the object distance. (b) Simplify the expression in part (a) for the case in which the object distance is much greater than objective focal length. (c) The "wingspan" of the International Space Station is 108.6 m, the overall width of its solar-panel configuration. Find the width of the image formed by a telescope objective of focal length 4.00 m when the station is orbiting at an altitude of 407 km.

53. Galileo devised a simple terrestrial telescope that produces an upright image. It consists of a converging objective lens and a diverging eyepiece at opposite ends of the telescope tube. For distant objects, the tube length is the objective focal length less the absolute value of the eyepiece focal length. (a) Does the user of the telescope see a real or virtual image? (b) Where is the final image? (c) If a telescope is to be constructed with a tube 10.0 cm long and a magnification of 3.00, what are the focal lengths of the objective and eyepiece?

54. A certain telescope has an objective mirror with an aperture diameter of 200 mm and a focal length of 2 000 mm. It captures the image of a nebula on photographic film at its prime focus with an exposure time of 1.50 min. To produce the same light energy per unit area on the film, what is the required exposure time to photograph the same nebula with a smaller telescope, which has an objective lens with an aperture diameter of 60.0 mm and a focal length of 900 mm?

ADDITIONAL PROBLEMS

55. The distance between an object and its upright image is 20.0 cm. If the magnification is 0.500, what is the focal length of the lens that is being used to form the image?

56. The distance between an object and its upright image is d. If the magnification is M, what is the focal length of the lens that is being used to form the image?

57. The lens and mirror in Figure P36.57 have focal lengths of $+80.0$ cm and -50.0 cm, respectively. An object is

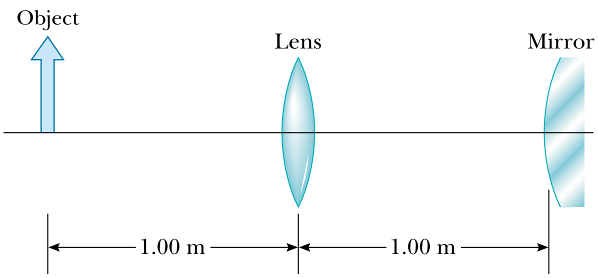


Figure P36.57

placed 1.00 m to the left of the lens, as shown. Locate the final image, which is formed by light that has gone through the lens twice. State whether the image is upright or inverted, and determine the overall magnification.

58. Your friend needs glasses with diverging lenses of focal length -65.0 cm for both eyes. You tell him he looks good when he does not squint, but he is worried about how thick the lenses will be. If the radius of curvature of the first surface is $R_1 = 50.0$ cm and the high-index plastic has a refractive index of 1.66, (a) find the required radius of curvature of the second surface. (b) Assume that the lens is ground from a disk 4.00 cm in diameter and 0.100 cm thick at the center. Find the thickness of the plastic at the edge of the lens, measured parallel to the axis. *Hint:* Draw a large cross-sectional diagram.

59. The object in Figure P36.59 is midway between the lens and the mirror. The mirror's radius of curvature is 20.0 cm, and the lens has a focal length of -16.7 cm. Considering only the light that leaves the object and travels first toward the mirror, locate the final image formed by this system. Is this image real or virtual? Is it upright or inverted? What is the overall magnification?

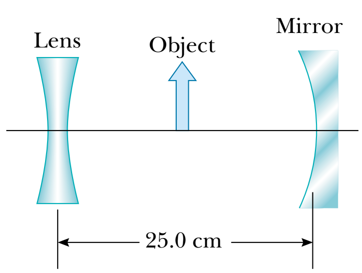


Figure P36.59

60. An object placed 10.0 cm from a concave spherical mirror produces a real image 8.00 cm from the mirror. If the object is moved to a new position 20.0 cm from the mirror, what is the position of the image? Is the latter image real or virtual?

WEB **61.** A parallel beam of light enters a glass hemisphere perpendicular to the flat face, as shown in Figure P36.61. The radius is $|R| = 6.00$ cm, and the index of refraction is $n = 1.560$. Determine the point at which the beam is focused. (Assume paraxial rays.)

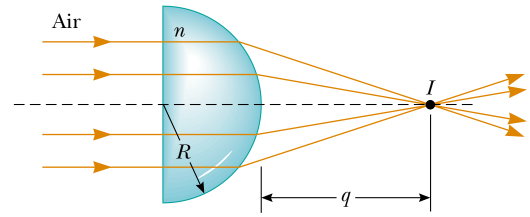


Figure P36.61

62. Review Problem. A spherical lightbulb with a diameter of 3.20 cm radiates light equally in all directions, with a power of 4.50 W. (a) Find the light intensity at the surface of the bulb. (b) Find the light intensity 7.20 m from the center of the bulb. (c) At this 7.20-m distance, a lens is set up with its axis pointing toward the bulb. The lens has a circular face with a diameter of 15.0 cm and a focal length of 35.0 cm. Find the diameter of the image of the bulb. (d) Find the light intensity at the image.

63. An object is placed 12.0 cm to the left of a diverging lens with a focal length of -6.00 cm. A converging lens with a focal length of 12.0 cm is placed a distance d to the right of the diverging lens. Find the distance d that corresponds to a final image at infinity. Draw a ray diagram for this case.

64. Assume that the intensity of sunlight is 1.00 kW/m^2 at a particular location. A highly reflecting concave mirror is to be pointed toward the Sun to produce a power of at least 350 W at the image. (a) Find the required radius R_a of the circular face area of the mirror. (b) Now suppose the light intensity is to be at least 120 kW/m^2 at the image. Find the required relationship between R_a and the radius of curvature R of the mirror. The disk of the Sun subtends an angle of 0.533° at the Earth.

WEB **65.** The disk of the Sun subtends an angle of 0.533° at the Earth. What are the position and diameter of the solar image formed by a concave spherical mirror with a radius of curvature of 3.00 m?

66. Figure P36.66 shows a thin converging lens for which the radii are $R_1 = 9.00$ cm and $R_2 = -11.0$ cm. The lens is in front of a concave spherical mirror of radius $R = 8.00$ cm. (a) If its focal points F_1 and F_2 are 5.00 cm from the vertex of the lens, determine its index of refraction. (b) If the lens and mirror are 20.0 cm apart and an object is placed 8.00 cm to the left of the

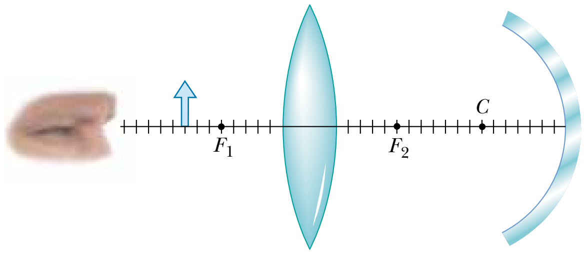


Figure P36.66

lens, determine the position of the final image and its magnification as seen by the eye in the figure. (c) Is the final image inverted or upright? Explain.

67. In a darkened room, a burning candle is placed 1.50 m from a white wall. A lens is placed between candle and wall at a location that causes a larger, inverted image to form on the wall. When the lens is moved 90.0 cm toward the wall, another image of the candle is formed. Find (a) the two object distances that produce the specified images and (b) the focal length of the lens. (c) Characterize the second image.

68. A thin lens of focal length f lies on a horizontal front-surfaced flat mirror. How far above the lens should an object be held if its image is to coincide with the object?

69. A compound microscope has an objective of focal length 0.300 cm and an eyepiece of focal length 2.50 cm. If an object is 3.40 mm from the objective, what is the magnification? (*Hint:* Use the lens equation for the objective.)

70. Two converging lenses with focal lengths of 10.0 cm and 20.0 cm are positioned 50.0 cm apart, as shown in Figure P36.70. The final image is to be located between the lenses, at the position indicated. (a) How far to the left of the first lens should the object be? (b) What is the overall magnification? (c) Is the final image upright or inverted?

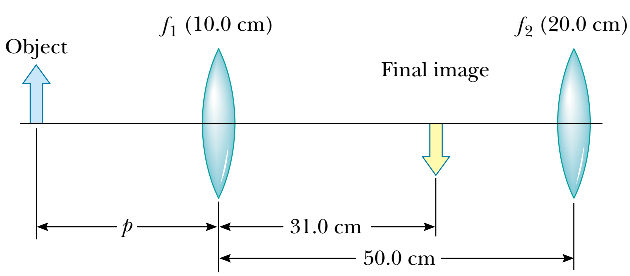


Figure P36.70

71. A cataract-impaired lens in an eye may be surgically removed and replaced by a manufactured lens. The focal length required for the new lens is determined by the lens-to-retina distance, which is measured by a sonar-like device, and by the requirement that the implant provide for correct distant vision. (a) If the distance from lens to retina is 22.4 mm, calculate the power of the implanted lens in diopters. (b) Since no accommodation occurs and the implant allows for correct distant vision, a corrective lens for close work or reading must be used. Assume a reading distance of 33.0 cm, and calculate the power of the lens in the reading glasses.

72. A floating strawberry illusion consists of two parabolic mirrors, each with a focal length of 7.50 cm, facing each other so that their centers are 7.50 cm apart (Fig. P36.72). If a strawberry is placed on the lower mirror, an image of the strawberry is formed at the small opening at the center of the top mirror. Show that the final image is formed at that location, and describe its characteristics. (*Note:* A very startling effect is to shine a flashlight beam on these images. Even at a glancing angle, the incoming light beam is seemingly reflected off the images! Do you understand why?)

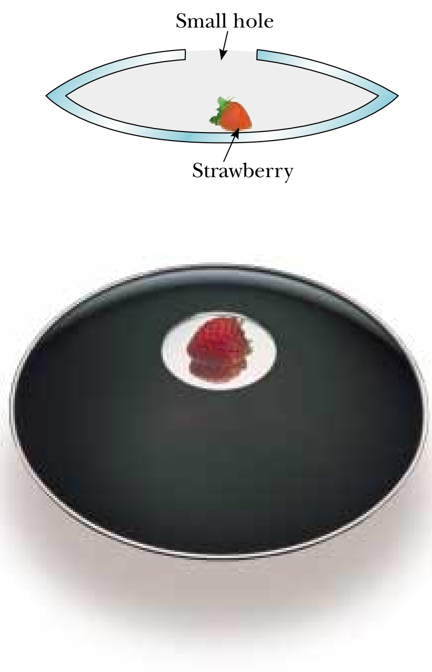


Figure P36.72

73. An object 2.00 cm high is placed 40.0 cm to the left of a converging lens with a focal length of 30.0 cm. A diverging lens with a focal length of -20.0 cm is placed 110 cm to the right of the converging lens. (a) Determine the final position and magnification of the final image. (b) Is the image upright or inverted? (c) Repeat parts (a) and (b) for the case in which the second lens is a converging lens with a focal length of $+20.0$ cm.

Answers to Quick Quizzes

36.1 At C. A ray traced from the stone to the mirror and then to observer 2 looks like this:

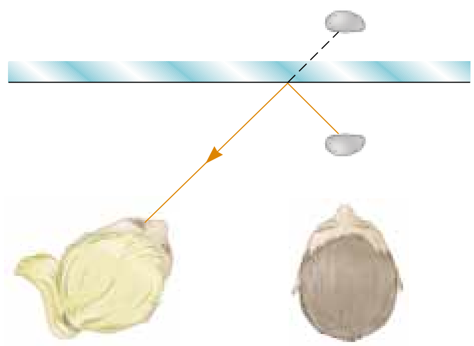


Figure QQA36.1

36.2 The focal length is infinite. Because the flat surfaces of the pane have infinite radii of curvature, Equation 36.11 indicates that the focal length is also infinite. Parallel rays striking the pane focus at infinity, which means that they remain parallel after passing through the glass.

36.3 An infinite number. In general, an infinite number of rays leave each point of any object and travel outward in all directions. (The three principal rays that we use to locate an image make up a selected subset of the infinite number of rays.) When an object is taller than a lens, we merely extend the plane containing the lens, as shown in Figure QQA36.2.

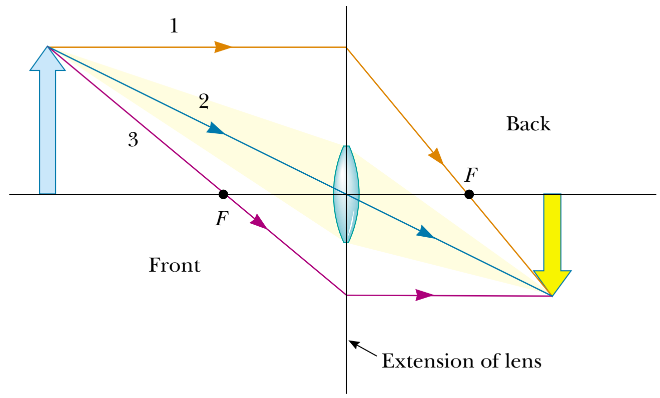
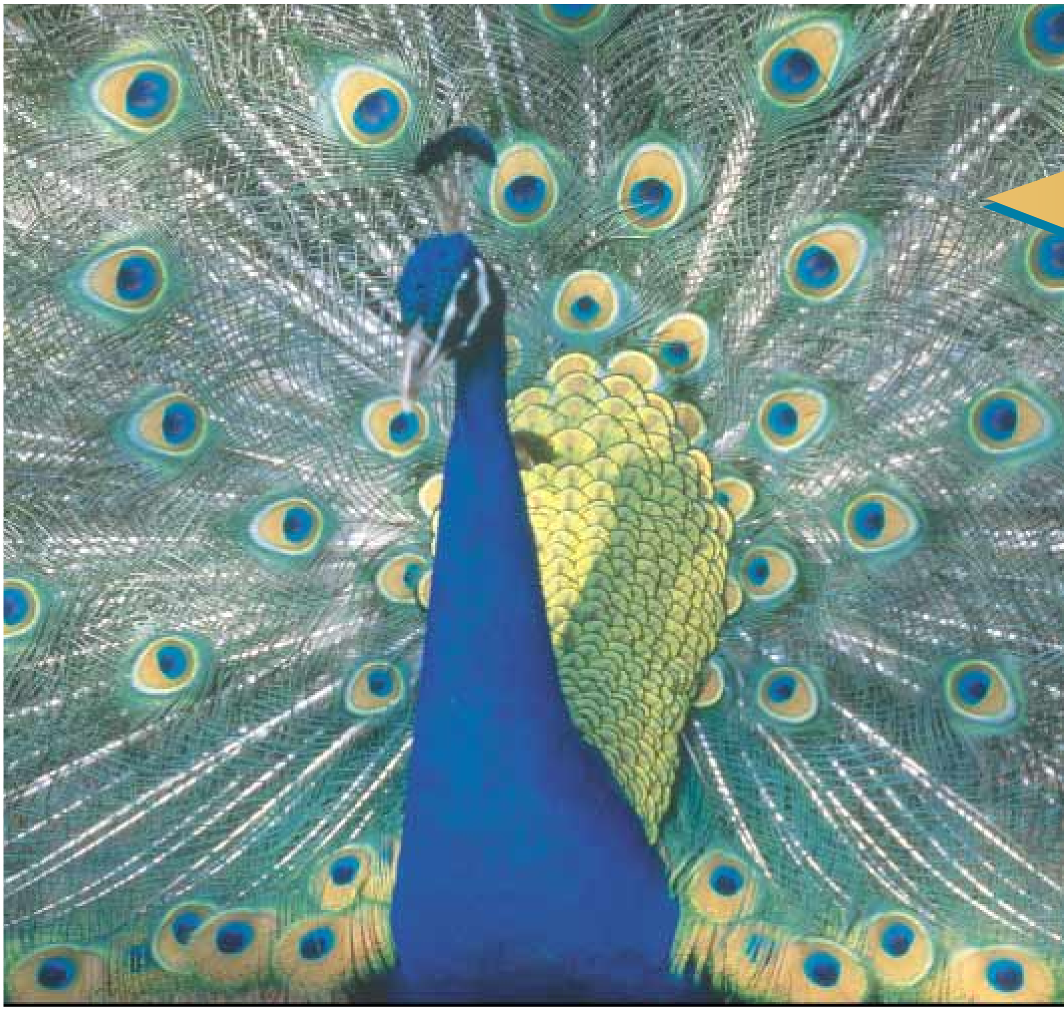


Figure QQA36.2

36.4 (c) The entire image is visible but has half the intensity. Each point on the object is a source of rays that travel in all directions. Thus, light from all parts of the object goes through all parts of the lens and forms an image. If you block part of the lens, you are blocking some of the rays, but the remaining ones still come from all parts of the object.

36.5 The eyeglasses on the left are diverging lenses, which correct for nearsightedness. If you look carefully at the edge of the person's face through the lens, you will see that everything viewed through these glasses is reduced in size. The eyeglasses on the right are converging lenses, which correct for farsightedness. These lenses make everything that is viewed through them look larger.

36.6 The lateral magnification of a telescope is not well defined. For viewing with the eye relaxed, the user may slightly adjust the position of the eyepiece to place the final image I_2 in Figure 36.42a at infinity. Then, its height and its lateral magnification also are infinite. The angular magnification of a telescope as we define it is the factor by which the telescope increases in the diameter—on the retina of the viewer's eye—of the real image of an extended object.



PUZZLER

The brilliant colors seen in peacock feathers are not caused by pigments in the feathers. If they are not produced by pigments, how *are* these beautiful colors created? *(Terry Qing/FPG International)*

chapter 37

Interference of Light Waves

Chapter Outline

37.1 Conditions for Interference

37.2 Young's Double-Slit Experiment

37.3 Intensity Distribution of the Double-Slit Interference Pattern

37.4 Phasor Addition of Waves

37.5 Change of Phase Due to Reflection

37.6 Interference in Thin Films

37.7 *(Optional)* The Michelson Interferometer

In the preceding chapter on geometric optics, we used light rays to examine what happens when light passes through a lens or reflects from a mirror. Here in Chapter 37 and in the next chapter, we are concerned with *wave optics,* the study of interference, diffraction, and polarization of light. These phenomena cannot be adequately explained with the ray optics used in Chapter 36. We now learn how treating light as waves rather than as rays leads to a satisfying description of such phenomena.

37.1 CONDITIONS FOR INTERFERENCE

In Chapter 18, we found that the adding together of two mechanical waves can be constructive or destructive. In constructive interference, the amplitude of the resultant wave is greater than that of either individual wave, whereas in destructive interference, the resultant amplitude is less than that of either individual wave. Light waves also interfere with each other. Fundamentally, all interference associated with light waves arises when the electromagnetic fields that constitute the individual waves combine.

If two lightbulbs are placed side by side, no interference effects are observed because the light waves from one bulb are emitted independently of those from the other bulb. The emissions from the two lightbulbs do not maintain a constant phase relationship with each other over time. Light waves from an ordinary source such as a lightbulb undergo random changes about once every 10^{-8} s. Therefore, the conditions for constructive interference, destructive interference, or some intermediate state last for lengths of time of the order of 10^{-8} s. Because the eye cannot follow such short-term changes, no interference effects are observed. (In 1993 interference from two separate light sources was photographed in an extremely fast exposure. Nonetheless, we do not ordinarily see interference effects because of the rapidly changing phase relationship between the light waves.) Such light sources are said to be **incoherent.**

Interference effects in light waves are not easy to observe because of the short wavelengths involved (from 4×10^{-7} m to 7×10^{-7} m). For sustained interference in light waves to be observed, the following conditions must be met:

Conditions for interference

- The sources must be **coherent**—that is, they must maintain a constant phase with respect to each other.
- The sources should be **monochromatic**—that is, of a single wavelength.

We now describe the characteristics of coherent sources. As we saw when we studied mechanical waves, two sources (producing two traveling waves) are needed to create interference. In order to produce a stable interference pattern, **the individual waves must maintain a constant phase relationship with one another.** As an example, the sound waves emitted by two side-by-side loudspeakers driven by a single amplifier can interfere with each other because the two speakers are coherent—that is, they respond to the amplifier in the same way at the same time.

A common method for producing two coherent light sources is to use one monochromatic source to illuminate a barrier containing two small openings (usually in the shape of slits). The light emerging from the two slits is coherent because a single source produces the original light beam and the two slits serve only to separate the original beam into two parts (which, after all, is what was done to the sound signal from the side-by-side loudspeakers). Any random change in the light

emitted by the source occurs in both beams at the same time, and as a result interference effects can be observed when the light from the two slits arrives at a viewing screen.

37.2 YOUNG'S DOUBLE-SLIT EXPERIMENT

Interference in light waves from two sources was first demonstrated by Thomas Young in 1801. A schematic diagram of the apparatus that Young used is shown in Figure 37.1a. Light is incident on a first barrier in which there is a slit S_0. The waves emerging from this slit arrive at a second barrier that contains two parallel slits S_1 and S_2. These two slits serve as a pair of coherent light sources because waves emerging from them originate from the same wave front and therefore maintain a constant phase relationship. The light from S_1 and S_2 produces on a viewing screen a visible pattern of bright and dark parallel bands called **fringes** (Fig. 37.1b). When the light from S_1 and that from S_2 both arrive at a point on the screen such that constructive interference occurs at that location, a bright fringe appears. When the light from the two slits combines destructively at any location on the screen, a dark fringe results. Figure 37.2 is a photograph of an interference pattern produced by two coherent vibrating sources in a water tank.

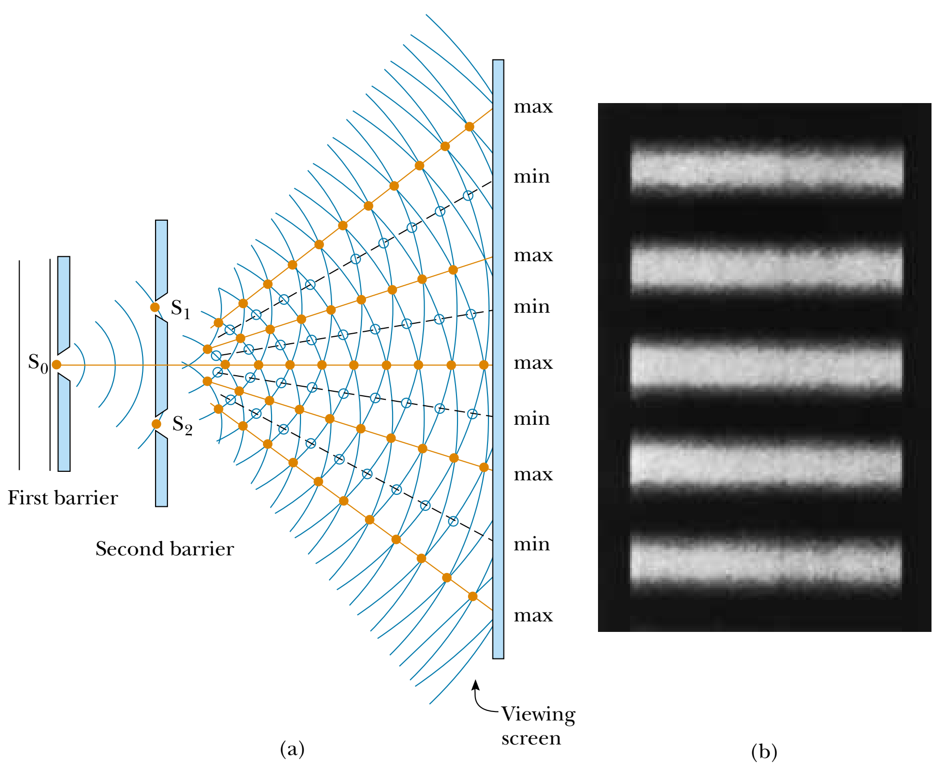


Figure 37.1 (a) Schematic diagram of Young's double-slit experiment. Slits S_1 and S_2 behave as coherent sources of light waves that produce an interference pattern on the viewing screen (drawing not to scale). (b) An enlargement of the center of a fringe pattern formed on the viewing screen with many slits could look like this.

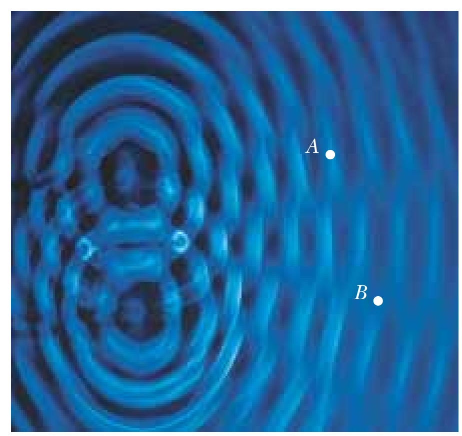


Figure 37.2 An interference pattern involving water waves is produced by two vibrating sources at the water's surface. The pattern is analogous to that observed in Young's double-slit experiment. Note the regions of constructive (A) and destructive (B) interference.

Quick Quiz 37.1

If you were to blow smoke into the space between the second barrier and the viewing screen of Figure 37.1a, what would you see?

QuickLab

Look through the fabric of an umbrella at a distant streetlight. Can you explain what you see? (The fringe pattern in Figure 37.1b is from rectangular slits. The fabric of the umbrella creates a two-dimensional set of square holes.)

Quick Quiz 37.2

Figure 37.2 is an overhead view of a shallow water tank. If you wanted to use a small ruler to measure the water's depth, would this be easier to do at location *A* or at location *B*?

Figure 37.3 shows some of the ways in which two waves can combine at the screen. In Figure 37.3a, the two waves, which leave the two slits in phase, strike the screen at the central point *P*. Because both waves travel the same distance, they arrive at *P* in phase. As a result, constructive interference occurs at this location, and a bright fringe is observed. In Figure 37.3b, the two waves also start in phase, but in this case the upper wave has to travel one wavelength farther than the lower wave to reach point *Q*. Because the upper wave falls behind the lower one by exactly one wavelength, they still arrive in phase at *Q*, and so a second bright fringe appears at this location. At point *R* in Figure 37.3c, however, midway between points *P* and *Q*, the upper wave has fallen half a wavelength behind the lower wave. This means that a trough of the lower wave overlaps a crest of the upper wave; this gives rise to destructive interference at point *R*. For this reason, a dark fringe is observed at this location.

We can describe Young's experiment quantitatively with the help of Figure 37.4. The viewing screen is located a perpendicular distance *L* from the double-slitted barrier. S_1 and S_2 are separated by a distance *d*, and the source is monochromatic. To reach any arbitrary point *P*, a wave from the lower slit travels farther than a wave from the upper slit by a distance $d \sin \theta$. This distance is called the **path difference** δ (lowercase Greek delta). If we assume that r_1 and r_2 are parallel, which is approximately true because *L* is much greater than *d*, then δ is given by

Path difference

$$\delta = r_2 - r_1 = d \sin \theta \tag{37.1}$$

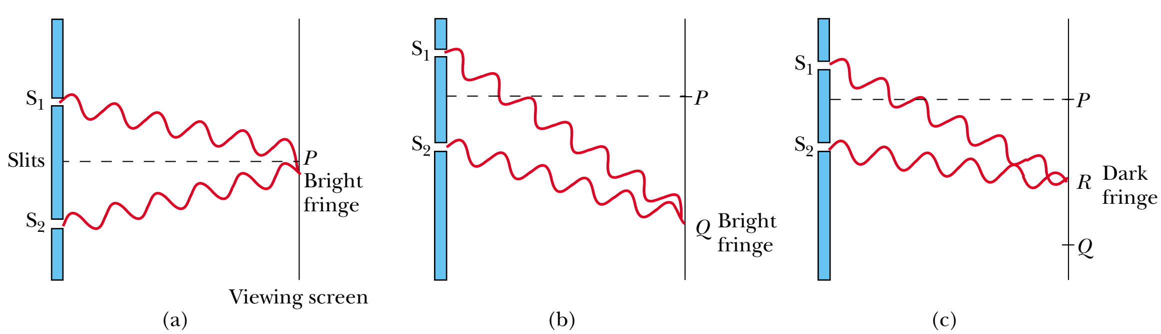


Figure 37.3 (a) Constructive interference occurs at point *P* when the waves combine. (b) Constructive interference also occurs at point *Q*. (c) Destructive interference occurs at *R* when the two waves combine because the upper wave falls half a wavelength behind the lower wave (all figures not to scale).

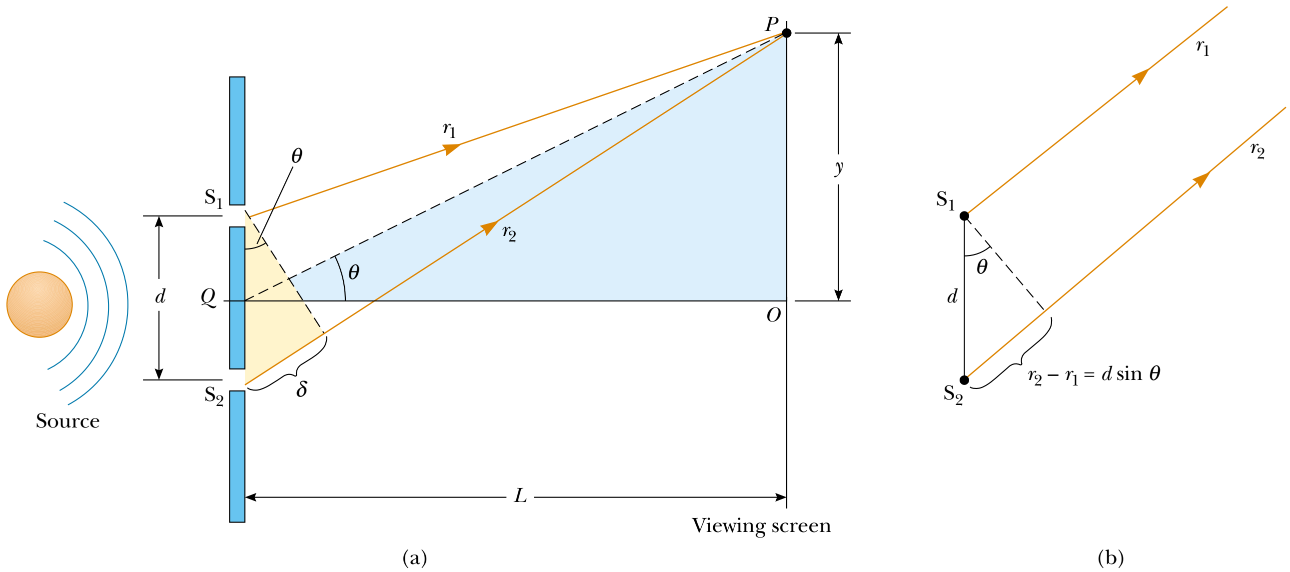


Figure 37.4 (a) Geometric construction for describing Young's double-slit experiment (not to scale). (b) When we assume that r_1 is parallel to r_2, the path difference between the two rays is $r_2 - r_1 = d \sin \theta$. For this approximation to be valid, it is essential that $L \gg d$.

The value of δ determines whether the two waves are in phase when they arrive at point P. If δ is either zero or some integer multiple of the wavelength, then the two waves are in phase at point P and constructive interference results. Therefore, the condition for bright fringes, or **constructive interference,** at point P is

Conditions for constructive interference

$$\delta = d \sin \theta = m\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \tag{37.2}$$

The number m is called the **order number.** The central bright fringe at $\theta = 0$ ($m = 0$) is called the *zeroth-order maximum.* The first maximum on either side, where $m = \pm 1$, is called the *first-order maximum,* and so forth.

When δ is an odd multiple of $\lambda/2$, the two waves arriving at point P are 180° out of phase and give rise to destructive interference. Therefore, the condition for dark fringes, or **destructive interference,** at point P is

Conditions for destructive interference

$$d \sin \theta = (m + \tfrac{1}{2})\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \tag{37.3}$$

It is useful to obtain expressions for the positions of the bright and dark fringes measured vertically from O to P. In addition to our assumption that $L \gg d$, we assume that $d \gg \lambda$. These can be valid assumptions because in practice L is often of the order of 1 m, d a fraction of a millimeter, and λ a fraction of a micrometer for visible light. Under these conditions, θ is small; thus, we can use the approximation $\sin \theta \approx \tan \theta$. Then, from triangle OPQ in Figure 37.4, we see that

$$y = L \tan \theta \approx L \sin \theta \tag{37.4}$$

Solving Equation 37.2 for $\sin \theta$ and substituting the result into Equation 37.4, we see that the positions of the bright fringes measured from O are given by the expression

$$y_{\text{bright}} = \frac{\lambda L}{d} m \tag{37.5}$$

Using Equations 37.3 and 37.4, we find that the dark fringes are located at

$$y_{\text{dark}} = \frac{\lambda L}{d}\left(m + \tfrac{1}{2}\right) \tag{37.6}$$

As we demonstrate in Example 37.1, Young's double-slit experiment provides a method for measuring the wavelength of light. In fact, Young used this technique to do just that. Additionally, the experiment gave the wave model of light a great deal of credibility. It was inconceivable that particles of light coming through the slits could cancel each other in a way that would explain the dark fringes.

EXAMPLE 37.1 Measuring the Wavelength of a Light Source

A viewing screen is separated from a double-slit source by 1.2 m. The distance between the two slits is 0.030 mm. The second-order bright fringe ($m = 2$) is 4.5 cm from the center line. (a) Determine the wavelength of the light.

Solution We can use Equation 37.5, with $m = 2$, $y_2 = 4.5 \times 10^{-2}$ m, $L = 1.2$ m, and $d = 3.0 \times 10^{-5}$ m:

$$\lambda = \frac{dy_2}{mL} = \frac{(3.0 \times 10^{-5}\ \text{m})(4.5 \times 10^{-2}\ \text{m})}{2(1.2\ \text{m})}$$

$$= 5.6 \times 10^{-7}\ \text{m} = \boxed{560\ \text{nm}}$$

(b) Calculate the distance between adjacent bright fringes.

Solution From Equation 37.5 and the results of part (a), we obtain

$$y_{m+1} - y_m = \frac{\lambda L(m+1)}{d} - \frac{\lambda L m}{d}$$

$$= \frac{\lambda L}{d} = \frac{(5.6 \times 10^{-7}\ \text{m})(1.2\ \text{m})}{3.0 \times 10^{-5}\ \text{m}}$$

$$= 2.2 \times 10^{-2}\ \text{m} = \boxed{2.2\ \text{cm}}$$

Note that the spacing between all fringes is equal.

EXAMPLE 37.2 Separating Double-Slit Fringes of Two Wavelengths

A light source emits visible light of two wavelengths: $\lambda = 430$ nm and $\lambda' = 510$ nm. The source is used in a double-slit interference experiment in which $L = 1.5$ m and $d = 0.025$ mm. Find the separation distance between the third-order bright fringes.

Solution Using Equation 37.5, with $m = 3$, we find that the fringe positions corresponding to these two wavelengths are

$$y_3 = \frac{\lambda L}{d} m = 3\frac{\lambda L}{d} = 7.74 \times 10^{-2}\ \text{m}$$

$$y_3' = \frac{\lambda' L}{d} m = 3\frac{\lambda' L}{d} = 9.18 \times 10^{-2}\ \text{m}$$

Hence, the separation distance between the two fringes is

$$\Delta y = y_3' - y_3 = 9.18 \times 10^{-2}\ \text{m} - 7.74 \times 10^{-2}\ \text{m}$$

$$= 1.4 \times 10^{-2}\ \text{m} = \boxed{1.4\ \text{cm}}$$

37.3 INTENSITY DISTRIBUTION OF THE DOUBLE-SLIT INTERFERENCE PATTERN

Note that the edges of the bright fringes in Figure 37.1b are fuzzy. So far we have discussed the locations of only the centers of the bright and dark fringes on a distant screen. We now direct our attention to the intensity of the light at other points between the positions of maximum constructive and destructive interference. In other words, we now calculate the distribution of light intensity associated with the double-slit interference pattern.

Again, suppose that the two slits represent coherent sources of sinusoidal waves such that the two waves from the slits have the same angular frequency ω and a constant phase difference ϕ. The total magnitude of the electric field at point P on the screen in Figure 37.5 is the vector superposition of the two waves. Assuming that the two waves have the same amplitude E_0, we can write the magnitude of the electric field at point P due to each wave separately as

$$E_1 = E_0 \sin \omega t \quad \text{and} \quad E_2 = E_0 \sin(\omega t + \phi) \tag{37.7}$$

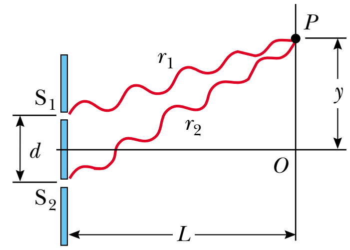


Figure 37.5 Construction for analyzing the double-slit interference pattern. A bright fringe, or intensity maximum, is observed at O.

Although the waves are in phase at the slits, *their phase difference ϕ at point P depends on the path difference $\delta = r_2 - r_1 = d \sin \theta$*. Because a path difference of λ (constructive interference) corresponds to a phase difference of 2π rad, we obtain the ratio

$$\frac{\delta}{\lambda} = \frac{\phi}{2\pi}$$

$$\phi = \frac{2\pi}{\lambda}\delta = \frac{2\pi}{\lambda} d \sin \theta \tag{37.8}$$

Phase difference

This equation tells us precisely how the phase difference ϕ depends on the angle θ in Figure 37.4.

Using the superposition principle and Equation 37.7, we can obtain the magnitude of the resultant electric field at point P:

$$E_P = E_1 + E_2 = E_0[\sin \omega t + \sin(\omega t + \phi)] \tag{37.9}$$

To simplify this expression, we use the trigonometric identity

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

Taking $A = \omega t + \phi$ and $B = \omega t$, we can write Equation 37.9 in the form

$$E_P = 2E_0 \cos\left(\frac{\phi}{2}\right) \sin\left(\omega t + \frac{\phi}{2}\right) \tag{37.10}$$

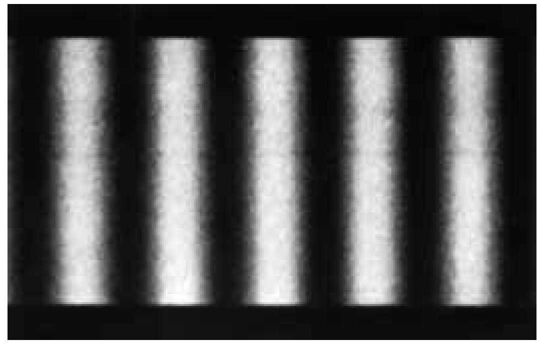
This result indicates that the electric field at point P has the same frequency ω as the light at the slits, but that the amplitude of the field is multiplied by the factor $2 \cos(\phi/2)$. To check the consistency of this result, note that if $\phi = 0, 2\pi, 4\pi, \ldots$, then the electric field at point P is $2E_0$, corresponding to the condition for constructive interference. These values of ϕ are consistent with Equation 37.2 for constructive interference. Likewise, if $\phi = \pi, 3\pi, 5\pi, \ldots$, then the magnitude of the electric field at point P is zero; this is consistent with Equation 37.3 for destructive interference.

Finally, to obtain an expression for the light intensity at point P, recall from Section 34.3 that *the intensity of a wave is proportional to the square of the resultant electric field magnitude at that point* (Eq. 34.20). Using Equation 37.10, we can therefore express the light intensity at point P as

$$I \propto E_P{}^2 = 4E_0{}^2 \cos^2\left(\frac{\phi}{2}\right) \sin^2\left(\omega t + \frac{\phi}{2}\right)$$

Most light-detecting instruments measure time-averaged light intensity, and the time-averaged value of $\sin^2(\omega t + \phi/2)$ over one cycle is $\frac{1}{2}$. Therefore, we can write the average light intensity at point P as

$$I = I_{\max} \cos^2\left(\frac{\phi}{2}\right) \tag{37.11}$$



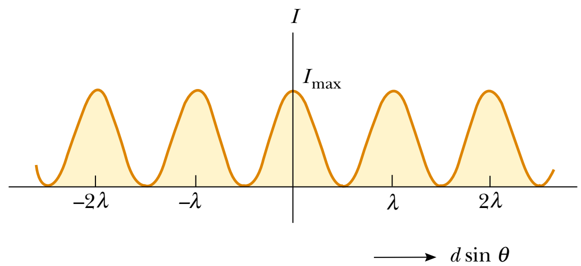


Figure 37.6 Light intensity versus $d \sin \theta$ for a double-slit interference pattern when the screen is far from the slits $(L \gg d)$.

where I_{max} is the maximum intensity on the screen and the expression represents the time average. Substituting the value for ϕ given by Equation 37.8 into this expression, we find that

$$I = I_{max} \cos^2\left(\frac{\pi d \sin \theta}{\lambda}\right) \quad \textbf{(37.12)}$$

Alternatively, because $\sin \theta \approx y/L$ for small values of θ in Figure 37.4, we can write Equation 37.12 in the form

$$I = I_{max} \cos^2\left(\frac{\pi d}{\lambda L}\, y\right) \quad \textbf{(37.13)}$$

Constructive interference, which produces light intensity maxima, occurs when the quantity $\pi dy/\lambda L$ is an integral multiple of π, corresponding to $y = (\lambda L/d)m$. This is consistent with Equation 37.5.

A plot of light intensity versus $d \sin \theta$ is given in Figure 37.6. Note that the interference pattern consists of equally spaced fringes of equal intensity. Remember, however, that this result is valid only if the slit-to-screen distance L is much greater than the slit separation, and only for small values of θ.

We have seen that the interference phenomena arising from two sources depend on the relative phase of the waves at a given point. Furthermore, the phase difference at a given point depends on the path difference between the two waves. **The resultant light intensity at a point is proportional to the square of the resultant electric field at that point.** That is, the light intensity is proportional to $(E_1 + E_2)^2$. It would be incorrect to calculate the light intensity by adding the intensities of the individual waves. This procedure would give $E_1{}^2 + E_2{}^2$, which of course is not the same as $(E_1 + E_2)^2$. Note, however, that $(E_1 + E_2)^2$ has the same *average* value as $E_1{}^2 + E_2{}^2$ when the time average is taken over all values of the

phase difference between E_1 and E_2. Hence, the law of conservation of energy is not violated.

37.4 PHASOR ADDITION OF WAVES

In the preceding section, we combined two waves algebraically to obtain the resultant wave amplitude at some point on a screen. Unfortunately, this analytical procedure becomes cumbersome when we must add several wave amplitudes. Because we shall eventually be interested in combining a large number of waves, we now describe a graphical procedure for this purpose.

Let us again consider a sinusoidal wave whose electric field component is given by

$$E_1 = E_0 \sin \omega t$$

where E_0 is the wave amplitude and ω is the angular frequency. This wave can be represented graphically by a phasor of magnitude E_0 rotating about the origin counterclockwise with an angular frequency ω, as shown in Figure 37.7a. Note that the phasor makes an angle ωt with the horizontal axis. The projection of the phasor on the vertical axis represents E_1, the magnitude of the wave disturbance at some time t. Hence, as the phasor rotates in a circle, the projection E_1 oscillates along the vertical axis about the origin.

Now consider a second sinusoidal wave whose electric field component is given by

$$E_2 = E_0 \sin(\omega t + \phi)$$

This wave has the same amplitude and frequency as E_1, but its phase is ϕ with respect to E_1. The phasor representing E_2 is shown in Figure 37.7b. We can obtain the resultant wave, which is the sum of E_1 and E_2, graphically by redrawing the phasors as shown in Figure 37.7c, in which the tail of the second phasor is placed at the tip of the first. As with vector addition, the resultant phasor $\mathbf{E}_R$ runs from the tail of the first phasor to the tip of the second. Furthermore, $\mathbf{E}_R$ rotates along with the two individual phasors at the same angular frequency ω. The projection of $\mathbf{E}_R$ along the vertical axis equals the sum of the projections of the two other phasors: $E_P = E_1 + E_2$.

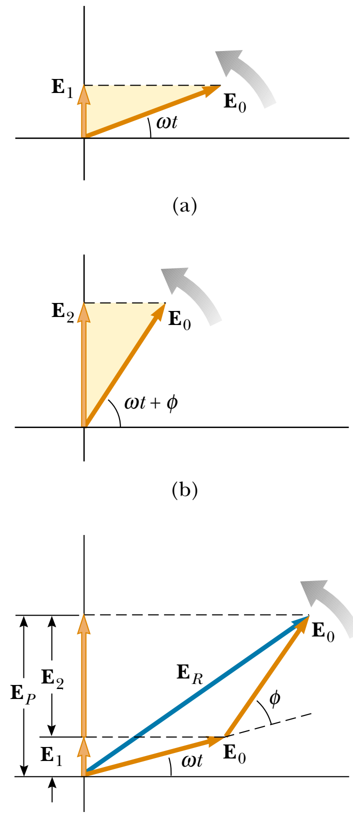


Figure 37.7 (a) Phasor diagram for the wave disturbance $E_1 = E_0 \sin \omega t$. The phasor is a vector of length E_0 rotating counterclockwise. (b) Phasor diagram for the wave $E_2 = E_0 \sin(\omega t + \phi)$. (c) The disturbance $\mathbf{E}_R$ is the resultant phasor formed from the phasors of parts (a) and (b).

It is convenient to construct the phasors at $t = 0$ as in Figure 37.8. From the geometry of one of the right triangles, we see that

$$\cos \alpha = \frac{E_R/2}{E_0}$$

which gives

$$E_R = 2E_0 \cos \alpha$$

Because the sum of the two opposite interior angles equals the exterior angle ϕ, we see that $\alpha = \phi/2$; thus,

$$E_R = 2E_0 \cos\left(\frac{\phi}{2}\right)$$

Hence, the projection of the phasor $\mathbf{E}_R$ along the vertical axis at any time t is

$$E_P = E_R \sin\left(\omega t + \frac{\phi}{2}\right) = 2E_0 \cos(\phi/2) \sin\left(\omega t + \frac{\phi}{2}\right)$$

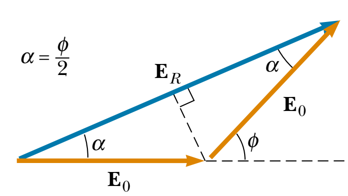


Figure 37.8 A reconstruction of the resultant phasor $\mathbf{E}_R$. From the geometry, note that $\alpha = \phi/2$.

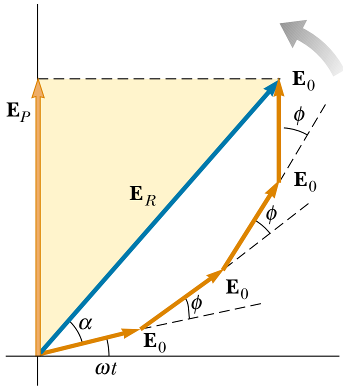


Figure 37.9 The phasor $\mathbf{E}_R$ is the resultant of four phasors of equal amplitude E_0. The phase of $\mathbf{E}_R$ with respect to the first phasor is α.

This is consistent with the result obtained algebraically, Equation 37.10. The resultant phasor has an amplitude $2E_0 \cos(\phi/2)$ and makes an angle $\phi/2$ with the first phasor. Furthermore, the average light intensity at point P, which varies as $E_P{}^2$, is proportional to $\cos^2(\phi/2)$, as described in Equation 37.11.

We can now describe how to obtain the resultant of several waves that have the same frequency:

- Represent the waves by phasors, as shown in Figure 37.9, remembering to maintain the proper phase relationship between one phasor and the next.
- The resultant phasor $\mathbf{E}_R$ is the vector sum of the individual phasors. At each instant, the projection of $\mathbf{E}_R$ along the vertical axis represents the time variation of the resultant wave. The phase angle α of the resultant wave is the angle between $\mathbf{E}_R$ and the first phasor. From Figure 37.9, drawn for four phasors, we see that the phasor of the resultant wave is given by the expression $E_P = E_R \sin(\omega t + \alpha)$.

Phasor Diagrams for Two Coherent Sources

As an example of the phasor method, consider the interference pattern produced by two coherent sources. Figure 37.10 represents the phasor diagrams for various values of the phase difference ϕ and the corresponding values of the path difference δ, which are obtained from Equation 37.8. The light intensity at a point is a maximum when $\mathbf{E}_R$ is a maximum; this occurs at $\phi = 0, 2\pi, 4\pi, \ldots$. The light intensity at some point is zero when $\mathbf{E}_R$ is zero; this occurs at $\phi = \pi, 3\pi, 5\pi, \ldots$. These results are in complete agreement with the analytical procedure described in the preceding section.

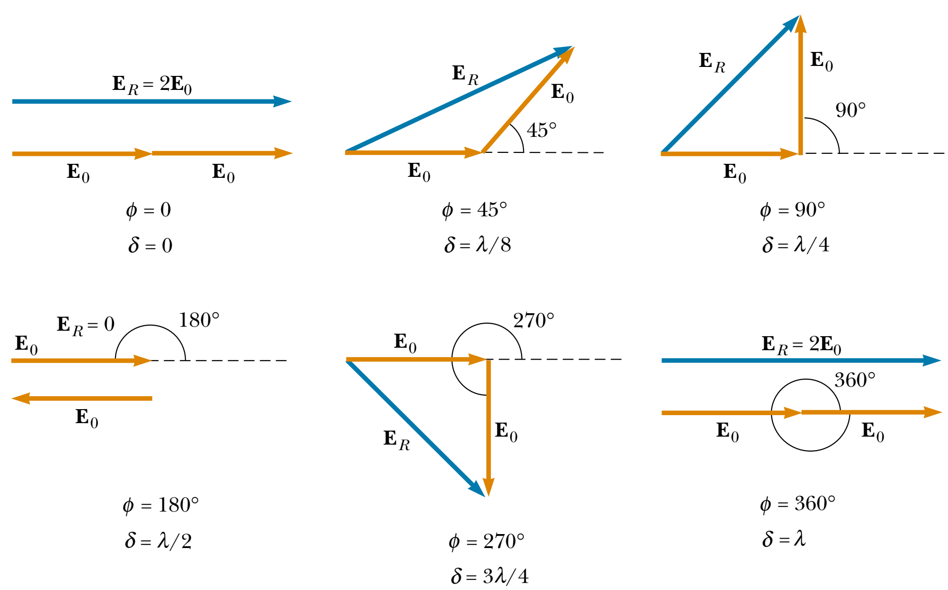


Figure 37.10 Phasor diagrams for a double-slit interference pattern. The resultant phasor $\mathbf{E}_R$ is a maximum when $\phi = 0, 2\pi, 4\pi, \ldots$ and is zero when $\phi = \pi, 3\pi, 5\pi, \ldots$.

Three-Slit Interference Pattern

Using phasor diagrams, let us analyze the interference pattern caused by three equally spaced slits. We can express the electric field components at a point P on the screen caused by waves from the individual slits as

$$E_1 = E_0 \sin \omega t$$

$$E_2 = E_0 \sin(\omega t + \phi)$$

$$E_3 = E_0 \sin(\omega t + 2\phi)$$

where ϕ is the phase difference between waves from adjacent slits. We can obtain the resultant magnitude of the electric field at point P from the phasor diagram in Figure 37.11.

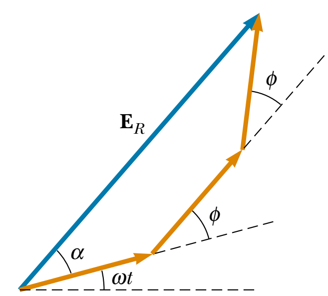


Figure 37.11 Phasor diagram for three equally spaced slits.

The phasor diagrams for various values of ϕ are shown in Figure 37.12. Note that the resultant magnitude of the electric field at P has a maximum value of $3E_0$, a condition that occurs when $\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$. These points are called *primary maxima.* Such primary maxima occur whenever the three phasors are aligned as shown in Figure 37.12a. We also find secondary maxima of amplitude E_0 occurring between the primary maxima at points where $\phi = \pm \pi, \pm 3\pi, \ldots$. For these points, the wave from one slit exactly cancels that from another slit (Fig. 37.12d). This means that only light from the third slit contributes to the resultant, which consequently has a total amplitude of E_0. Total destructive interference occurs whenever the three phasors form a closed triangle, as shown in Figure 37.12c. These points where $E_R = 0$ correspond to $\phi = \pm 2\pi/3, \pm 4\pi/3, \ldots$. You should be able to construct other phasor diagrams for values of ϕ greater than π.

Figure 37.13 shows multiple-slit interference patterns for a number of configurations. For three slits, note that the primary maxima are nine times more intense than the secondary maxima as measured by the height of the curve. This is because the intensity varies as E_R^2. For N slits, the intensity of the primary maxima is N^2 times greater than that due to a single slit. As the number of slits increases, the primary maxima increase in intensity and become narrower, while the secondary maxima decrease in intensity relative to the primary maxima. Figure 37.13 also shows that as the number of slits increases, the number of secondary maxima also increases. In fact, the number of secondary maxima is always $N - 2$, where N is the number of slits.

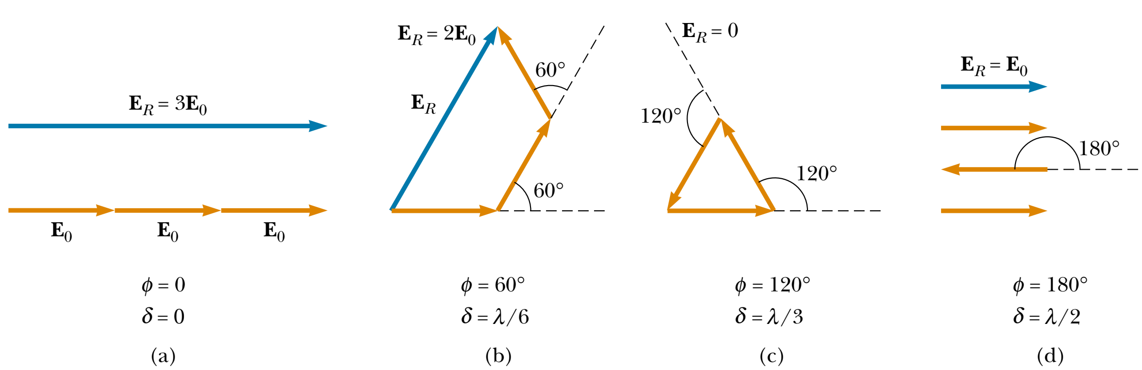


Figure 37.12 Phasor diagrams for three equally spaced slits at various values of ϕ. Note from (a) that there are primary maxima of amplitude $3E_0$ and from (d) that there are secondary maxima of amplitude E_0.

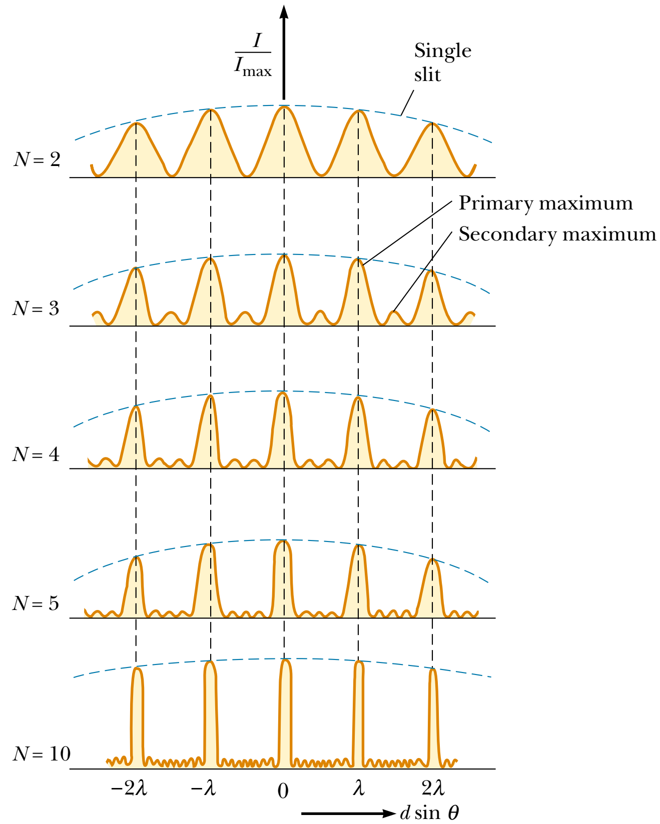


Figure 37.13 Multiple-slit interference patterns. As *N*, the number of slits, is increased, the primary maxima (the tallest peaks in each graph) become narrower but remain fixed in position, and the number of secondary maxima increases. For any value of *N*, the decrease in intensity in maxima to the left and right of the central maximum, indicated by the blue dashed arcs, is due to diffraction, which is discussed in Chapter 38.

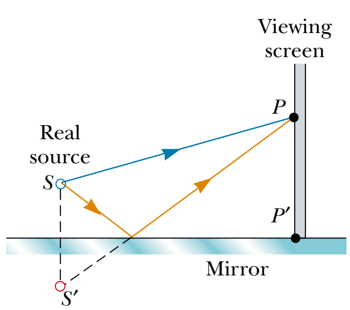


Figure 37.14 Lloyd's mirror. An interference pattern is produced at point *P* on the screen as a result of the combination of the direct ray (blue) and the reflected ray (red). The reflected ray undergoes a phase change of 180°.

Quick Quiz 37.3

Using Figure 37.13 as a model, sketch the interference pattern from six slits.

37.5 CHANGE OF PHASE DUE TO REFLECTION

Young's method for producing two coherent light sources involves illuminating a pair of slits with a single source. Another simple, yet ingenious, arrangement for producing an interference pattern with a single light source is known as *Lloyd's mirror* (Fig. 37.14). A light source is placed at point *S* close to a mirror, and a viewing screen is positioned some distance away at right angles to the mirror. Light waves can reach point *P* on the screen either by the direct path *SP* or by the path involving reflection from the mirror. The reflected ray can be treated as a ray originating from a virtual source at point *S*′. As a result, we can think of this arrangement as a double-slit source with the distance between

points S and S' comparable to length d in Figure 37.4. Hence, at observation points far from the source ($L \gg d$), we expect waves from points S and S' to form an interference pattern just like the one we see from two real coherent sources. An interference pattern is indeed observed. However, the positions of the dark and bright fringes are reversed relative to the pattern created by two real coherent sources (Young's experiment). This is because the coherent sources at points S and S' differ in phase by 180°, a phase change produced by reflection.

To illustrate this further, consider point P', the point where the mirror intersects the screen. This point is equidistant from points S and S'. If path difference alone were responsible for the phase difference, we would see a bright fringe at point P' (because the path difference is zero for this point), corresponding to the central bright fringe of the two-slit interference pattern. Instead, we observe a dark fringe at point P' because of the 180° phase change produced by reflection. In general,

an electromagnetic wave undergoes a phase change of 180° upon reflection from a medium that has a higher index of refraction than the one in which the wave is traveling.

It is useful to draw an analogy between reflected light waves and the reflections of a transverse wave pulse on a stretched string (see Section 16.6). The reflected pulse on a string undergoes a phase change of 180° when reflected from the boundary of a denser medium, but no phase change occurs when the pulse is reflected from the boundary of a less dense medium. Similarly, an electromagnetic wave undergoes a 180° phase change when reflected from a boundary leading to an optically denser medium, but no phase change occurs when the wave is reflected from a boundary leading to a less dense medium. These rules, summarized in Figure 37.15, can be deduced from Maxwell's equations, but the treatment is beyond the scope of this text.

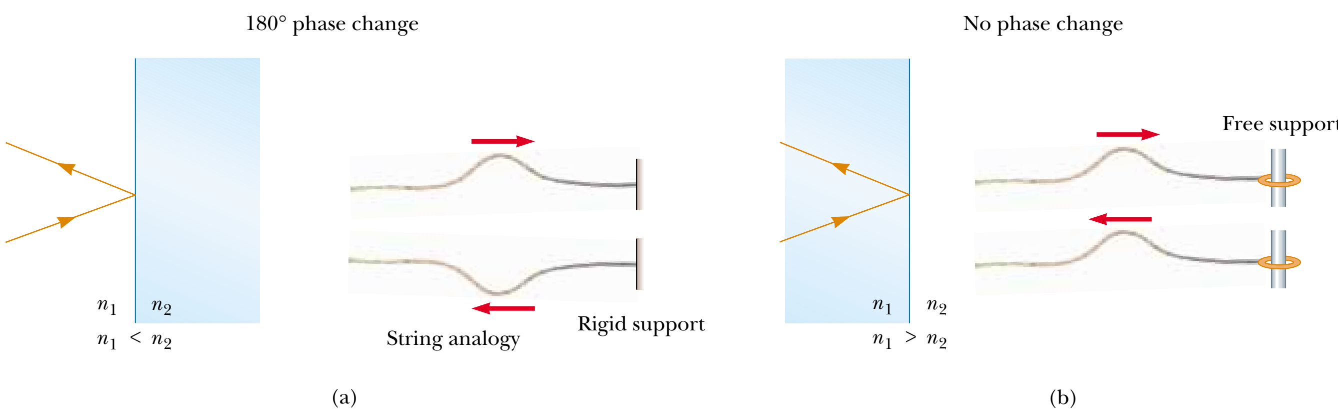


Figure 37.15 (a) For $n_1 < n_2$, a light ray traveling in medium 1 when reflected from the surface of medium 2 undergoes a 180° phase change. The same thing happens with a reflected pulse traveling along a string fixed at one end. (b) For $n_1 > n_2$, a light ray traveling in medium 1 undergoes no phase change when reflected from the surface of medium 2. The same is true of a reflected wave pulse on a string whose supported end is free to move.

37.6 INTERFERENCE IN THIN FILMS

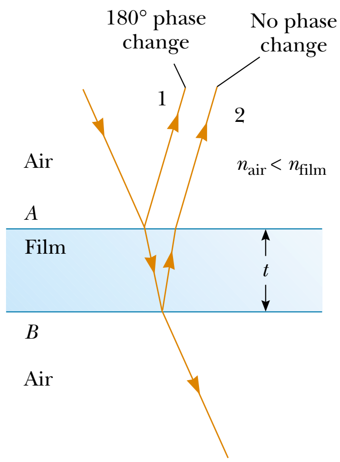


Figure 37.16 Interference in light reflected from a thin film is due to a combination of rays reflected from the upper and lower surfaces of the film.

Interference effects are commonly observed in thin films, such as thin layers of oil on water or the thin surface of a soap bubble. The varied colors observed when white light is incident on such films result from the interference of waves reflected from the two surfaces of the film.

Consider a film of uniform thickness t and index of refraction n, as shown in Figure 37.16. Let us assume that the light rays traveling in air are nearly normal to the two surfaces of the film. To determine whether the reflected rays interfere constructively or destructively, we first note the following facts:

- A wave traveling from a medium of index of refraction n_1 toward a medium of index of refraction n_2 undergoes a 180° phase change upon reflection when $n_2 > n_1$ and undergoes no phase change if $n_2 < n_1$.
- The wavelength of light λ_n in a medium whose refraction index is n (see Section 35.5) is

$$\lambda_n = \frac{\lambda}{n} \tag{37.14}$$

where λ is the wavelength of the light in free space.

Let us apply these rules to the film of Figure 37.16, where $n_{film} > n_{air}$. Reflected ray 1, which is reflected from the upper surface (A), undergoes a phase change of 180° with respect to the incident wave. Reflected ray 2, which is reflected from the lower film surface (B), undergoes no phase change because it is reflected from a medium (air) that has a lower index of refraction. Therefore, ray 1 is 180° out of phase with ray 2, which is equivalent to a path difference of $\lambda_n/2$.

Interference in soap bubbles. The colors are due to interference between light rays reflected from the front and back surfaces of the thin film of soap making up the bubble. The color depends on the thickness of the film, ranging from black where the film is thinnest to red where it is thickest.

The brilliant colors in a peacock's feathers are due to interference. The multilayer structure of the feathers causes constructive interference for certain colors, such as blue and green. The colors change as you view a peacock's feathers from different angles. Iridescent colors of butterflies and hummingbirds are the result of similar interference effects.

However, we must also consider that ray 2 travels an extra distance $2t$ before the waves recombine in the air above surface A. If $2t = \lambda_n/2$, then rays 1 and 2 recombine in phase, and the result is constructive interference. In general, the condition for constructive interference in such situations is

$$2t = (m + \tfrac{1}{2})\lambda_n \qquad m = 0, 1, 2, \ldots \tag{37.15}$$

This condition takes into account two factors: (1) the difference in path length for the two rays (the term $m\lambda_n$) and (2) the 180° phase change upon reflection (the term $\lambda_n/2$). Because $\lambda_n = \lambda/n$, we can write Equation 37.15 as

Conditions for constructive interference in thin films

$$2nt = (m + \tfrac{1}{2})\lambda \qquad m = 0, 1, 2, \ldots \tag{37.16}$$

If the extra distance $2t$ traveled by ray 2 corresponds to a multiple of λ_n, then the two waves combine out of phase, and the result is destructive interference. The general equation for destructive interference is

Conditions for destructive interference in thin films

$$2nt = m\lambda \qquad m = 0, 1, 2, \ldots \tag{37.17}$$

The foregoing conditions for constructive and destructive interference are valid when the medium above the top surface of the film is the same as the medium below the bottom surface. The medium surrounding the film may have a refractive index less than or greater than that of the film. In either case, the rays reflected from the two surfaces are out of phase by 180°. If the film is placed between two different media, one with $n < n_{\text{film}}$ and the other with $n > n_{\text{film}}$, then the conditions for constructive and destructive interference are reversed. In this case, either there is a phase change of 180° for both ray 1 reflecting from surface A and ray 2 reflecting from surface B, or there is no phase change for either ray; hence, the net change in relative phase due to the reflections is zero.

Quick Quiz 37.4

In Figure 37.17, where does the oil film thickness vary the least?

Figure 37.17 A thin film of oil floating on water displays interference, as shown by the pattern of colors produced when white light is incident on the film. Variations in film thickness produce the interesting color pattern. The razor blade gives one an idea of the size of the colored bands.

Newton's Rings

Another method for observing interference in light waves is to place a planoconvex lens on top of a flat glass surface, as shown in Figure 37.18a. With this arrangement, the air film between the glass surfaces varies in thickness from zero at the point of contact to some value t at point P. If the radius of curvature R of the lens is much greater than the distance r, and if the system is viewed from above using light of a single wavelength λ, a pattern of light and dark rings is observed, as shown in Figure 37.18b. These circular fringes, discovered by Newton, are called **Newton's rings.**

The interference effect is due to the combination of ray 1, reflected from the flat plate, with ray 2, reflected from the curved surface of the lens. Ray 1 undergoes a phase change of 180° upon reflection (because it is reflected from a medium of higher refractive index), whereas ray 2 undergoes no phase change (because it is reflected from a medium of lower refractive index). Hence, the conditions for constructive and destructive interference are given by Equations 37.16 and 37.17, respectively, with $n = 1$ because the film is air.

The contact point at O is dark, as seen in Figure 37.18b, because ray 1 undergoes a 180° phase change upon external reflection (from the flat surface); in con-

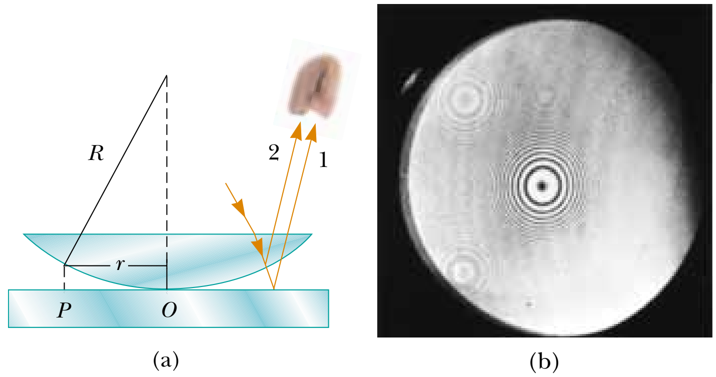


Figure 37.18 (a) The combination of rays reflected from the flat plate and the curved lens surface gives rise to an interference pattern known as Newton's rings. (b) Photograph of Newton's rings.

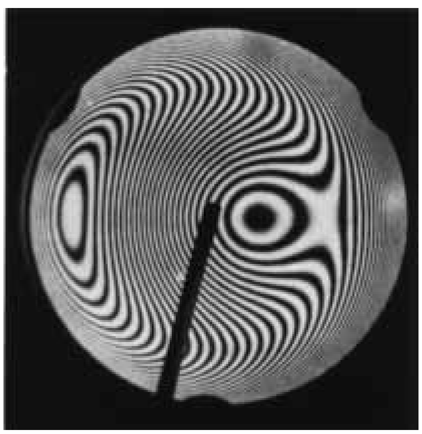


Figure 37.19 This asymmetrical interference pattern indicates imperfections in the lens of a Newton's-rings apparatus.

trast, ray 2 undergoes no phase change upon internal reflection (from the curved surface).

Using the geometry shown in Figure 37.18a, we can obtain expressions for the radii of the bright and dark bands in terms of the radius of curvature R and wavelength λ. For example, the dark rings have radii given by the expression $r \approx \sqrt{m\lambda R/n}$. The details are left as a problem for you to solve (see Problem 67). We can obtain the wavelength of the light causing the interference pattern by measuring the radii of the rings, provided R is known. Conversely, we can use a known wavelength to obtain R.

One important use of Newton's rings is in the testing of optical lenses. A circular pattern like that pictured in Figure 37.18b is obtained only when the lens is ground to a perfectly symmetric curvature. Variations from such symmetry might produce a pattern like that shown in Figure 37.19. These variations indicate how the lens must be reground and repolished to remove the imperfections.

Problem-Solving Hints

Thin-Film Interference

You should keep the following ideas in mind when you work thin-film interference problems:

- Identify the thin film causing the interference.
- The type of interference that occurs is determined by the phase relationship between the portion of the wave reflected at the upper surface of the film and the portion reflected at the lower surface.
- Phase differences between the two portions of the wave have two causes: (1) differences in the distances traveled by the two portions and (2) phase changes that may occur upon reflection.
- When the distance traveled and phase changes upon reflection are both taken into account, the interference is constructive if the equivalent path difference between the two waves is an integral multiple of λ, and it is destructive if the path difference is $\lambda/2$, $3\lambda/2$, $5\lambda/2$, and so forth.

QuickLab

Observe the colors appearing to swirl on the surface of a soap bubble. What do you see just before a bubble bursts? Why?

EXAMPLE 37.3 Interference in a Soap Film

Calculate the minimum thickness of a soap-bubble film ($n = 1.33$) that results in constructive interference in the reflected light if the film is illuminated with light whose wavelength in free space is $\lambda = 600$ nm.

Solution The minimum film thickness for constructive interference in the reflected light corresponds to $m = 0$ in Equation 37.16. This gives $2nt = \lambda/2$, or

$$t = \frac{\lambda}{4n} = \frac{600 \text{ nm}}{4(1.33)} = \boxed{113 \text{ nm}}$$

Exercise What other film thicknesses produce constructive interference?

Answer 338 nm, 564 nm, 789 nm, and so on.

EXAMPLE 37.4 Nonreflective Coatings for Solar Cells

Solar cells—devices that generate electricity when exposed to sunlight—are often coated with a transparent, thin film of silicon monoxide (SiO, $n = 1.45$) to minimize reflective losses from the surface. Suppose that a silicon solar cell ($n = 3.5$) is coated with a thin film of silicon monoxide for this purpose (Fig. 37.20). Determine the minimum film thickness that produces the least reflection at a wavelength of 550 nm, near the center of the visible spectrum.

Solution The reflected light is a minimum when rays 1 and 2 in Figure 37.20 meet the condition of destructive interference. Note that both rays undergo a 180° phase change upon reflection—ray 1 from the upper SiO surface and ray 2 from the lower SiO surface. The net change in phase due to reflection is therefore zero, and the condition for a reflection minimum requires a path difference of $\lambda_n/2$. Hence, $2t = \lambda/2n$, and the required thickness is

$$t = \frac{\lambda}{4n} = \frac{550 \text{ nm}}{4(1.45)} = \boxed{94.8 \text{ nm}}$$

A typical uncoated solar cell has reflective losses as high as 30%; a SiO coating can reduce this value to about 10%. This significant decrease in reflective losses increases the cell's efficiency because less reflection means that more sunlight enters the silicon to create charge carriers in the cell. No coating can ever be made perfectly nonreflecting because the required thickness is wavelength-dependent and the incident light covers a wide range of wavelengths.

Glass lenses used in cameras and other optical instruments are usually coated with a transparent thin film to reduce or eliminate unwanted reflection and enhance the transmission of light through the lenses.

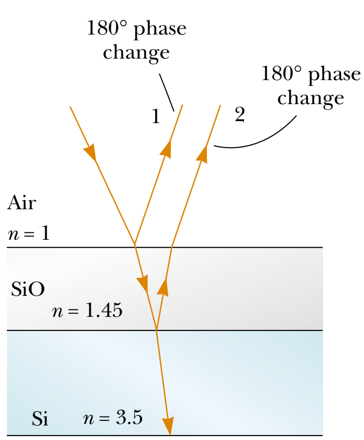


Figure 37.20 Reflective losses from a silicon solar cell are minimized by coating the surface of the cell with a thin film of silicon monoxide.

This camera lens has several coatings (of different thicknesses) that minimize reflection of light waves having wavelengths near the center of the visible spectrum. As a result, the little light that is reflected by the lens has a greater proportion of the far ends of the spectrum and appears reddish-violet.

EXAMPLE 37.5 Interference in a Wedge-Shaped Film

A thin, wedge-shaped film of refractive index n is illuminated with monochromatic light of wavelength λ, as illustrated in Figure 37.21a. Describe the interference pattern observed for this case.

Solution The interference pattern, because it is created by a thin film of variable thickness surrounded by air, is a series of alternating bright and dark parallel fringes. A dark fringe corresponding to destructive interference appears at point O, the apex, because here the upper reflected ray undergoes a 180° phase change while the lower one undergoes no phase change.

According to Equation 37.17, other dark minima appear when $2nt = m\lambda$; thus, $t_1 = \lambda/2n$, $t_2 = \lambda/n$, $t_3 = 3\lambda/2n$, and so on. Similarly, the bright maxima appear at locations where the thickness satisfies Equation 37.16, $2nt = (m + \frac{1}{2})\lambda$, corresponding to thicknesses of $\lambda/4n$, $3\lambda/4n$, $5\lambda/4n$, and so on.

If white light is used, bands of different colors are observed at different points, corresponding to the different wavelengths of light (see Fig. 37.21b). This is why we see different colors in soap bubbles.

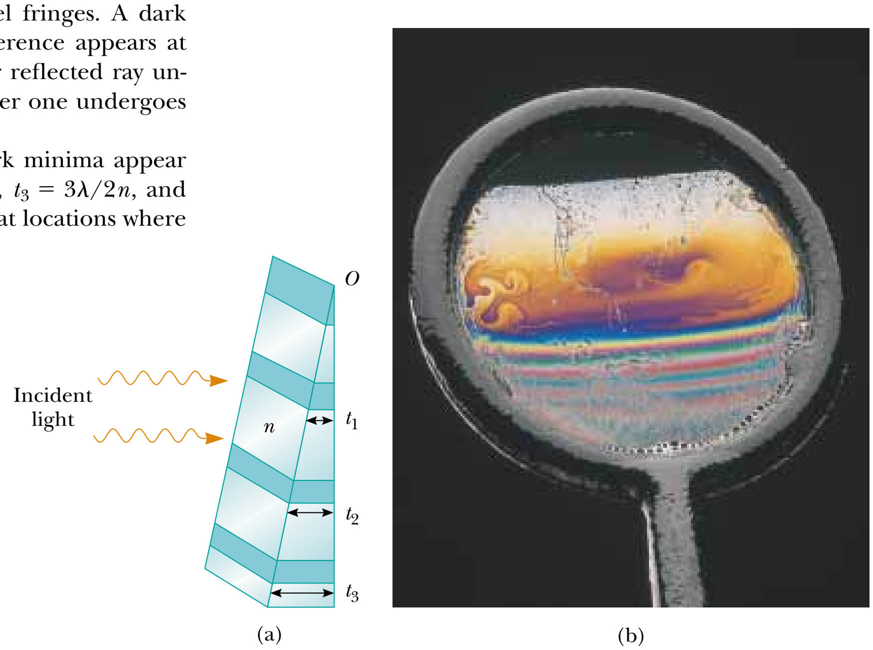


Figure 37.21 (a) Interference bands in reflected light can be observed by illuminating a wedge-shaped film with monochromatic light. The darker areas correspond to regions where rays tend to cancel each other because of interference effects. (b) Interference in a vertical film of variable thickness. The top of the film appears darkest where the film is thinnest.

Optional Section

37.7 THE MICHELSON INTERFEROMETER

The **interferometer,** invented by the American physicist A. A. Michelson (1852–1931), splits a light beam into two parts and then recombines the parts to form an interference pattern. The device can be used to measure wavelengths or other lengths with great precision.

A schematic diagram of the interferometer is shown in Figure 37.22. A ray of light from a monochromatic source is split into two rays by mirror M, which is inclined at 45° to the incident light beam. Mirror M, called a *beam splitter,* transmits half the light incident on it and reflects the rest. One ray is reflected from M vertically upward toward mirror M_1, and the second ray is transmitted horizontally through M toward mirror M_2. Hence, the two rays travel separate paths L_1 and L_2. After reflecting from M_1 and M_2, the two rays eventually recombine at M to produce an interference pattern, which can be viewed through a telescope. The glass plate P, equal in thickness to mirror M, is placed in the path of the horizontal ray to ensure that the two returning rays travel the same thickness of glass.

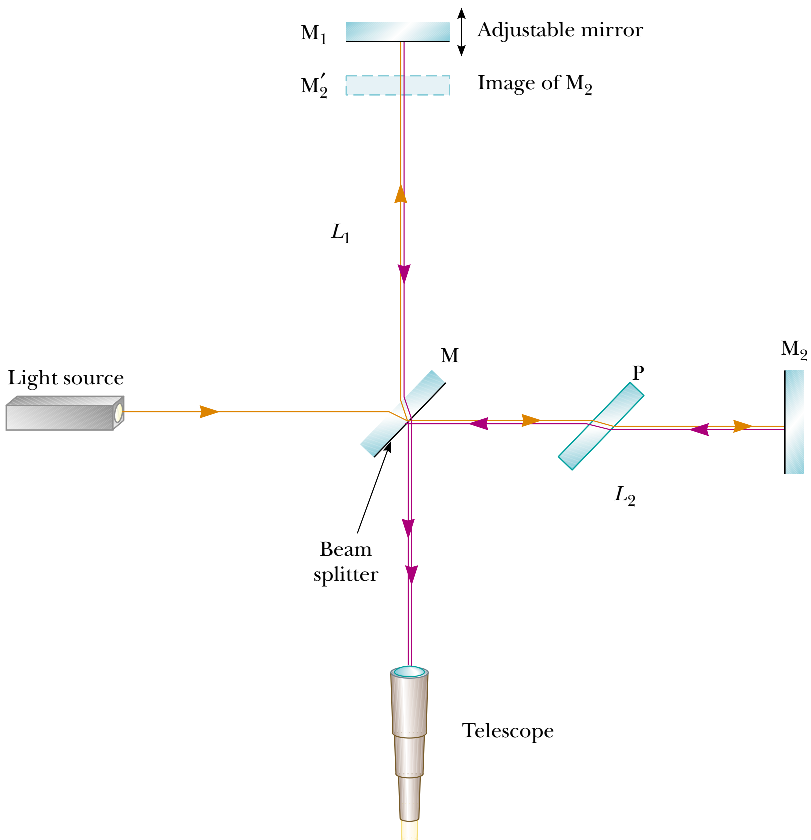


Figure 37.22 Diagram of the Michelson interferometer. A single ray of light is split into two rays by mirror M, which is called a beam splitter. The path difference between the two rays is varied with the adjustable mirror M_1. As M_1 is moved toward M, an interference pattern moves across the field of view.

The interference condition for the two rays is determined by their path length differences. When the two rays are viewed as shown, the image of M_2 produced by the mirror M is at M_2', which is nearly parallel to M_1. (Because M_1 and M_2 are not exactly perpendicular to each other, the image M_2' is at a slight angle to M_1.) Hence, the space between M_2' and M_1 is the equivalent of a wedge-shaped air film. The effective thickness of the air film is varied by moving mirror M_1 parallel to itself with a finely threaded screw adjustment. Under these conditions, the interference pattern is a series of bright and dark parallel fringes as described in Example 37.5. As M_1 is moved, the fringe pattern shifts. For example, if a dark fringe appears in the field of view (corresponding to destructive interference) and M_1 is then moved a distance $\lambda/4$ toward M, the path difference changes by $\lambda/2$ (twice the separation between M_1 and M_2'). What was a dark fringe now becomes a bright fringe. As M_1 is moved an additional distance $\lambda/4$ toward M, the bright fringe becomes a dark fringe. Thus, the fringe pattern shifts by one-half fringe each time M_1 is moved a distance $\lambda/4$. The wavelength of light is then measured by counting the number of fringe shifts for a given displacement of M_1. If the wavelength is accurately known (as with a laser beam), mirror displacements can be measured to within a fraction of the wavelength.

SUMMARY

Interference in light waves occurs whenever two or more waves overlap at a given point. A sustained interference pattern is observed if (1) the sources are coherent and (2) the sources have identical wavelengths.

In Young's double-slit experiment, two slits S_1 and S_2 separated by a distance d are illuminated by a single-wavelength light source. An interference pattern consisting of bright and dark fringes is observed on a viewing screen. The condition for bright fringes **(constructive interference)** is

$$d \sin \theta = m\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \tag{37.2}$$

The condition for dark fringes **(destructive interference)** is

$$d \sin \theta = (m + \tfrac{1}{2})\lambda \qquad m = 0, \pm 1, \pm 2, \ldots \tag{37.3}$$

The number m is called the **order number** of the fringe.

The **intensity** at a point in the double-slit interference pattern is

$$I = I_{\text{max}} \cos^2\left(\frac{\pi d \sin \theta}{\lambda}\right) \tag{37.12}$$

where I_{max} is the maximum intensity on the screen and the expression represents the time average.

A wave traveling from a medium of index of refraction n_1 toward a medium of index of refraction n_2 undergoes a 180° phase change upon reflection when $n_2 > n_1$ and undergoes no phase change when $n_2 < n_1$.

The condition for constructive interference in a film of thickness t and refractive index n surrounded by air is

$$2nt = (m + \tfrac{1}{2})\lambda \qquad m = 0, 1, 2, \ldots \tag{37.16}$$

where λ is the wavelength of the light in free space.

Similarly, the condition for destructive interference in a thin film is

$$2nt = m\lambda \qquad m = 0, 1, 2, \ldots \tag{37.17}$$

QUESTIONS

1. What is the necessary condition on the path length difference between two waves that interfere (a) constructively and (b) destructively?
2. Explain why two flashlights held close together do not produce an interference pattern on a distant screen.
3. If Young's double-slit experiment were performed under water, how would the observed interference pattern be affected?
4. In Young's double-slit experiment, why do we use monochromatic light? If white light is used, how would the pattern change?
5. Consider a dark fringe in an interference pattern, at which almost no light is arriving. Light from both slits is arriving at this point, but the waves are canceling. Where does the energy go?
6. An oil film on water appears brightest at the outer regions, where it is thinnest. From this information, what can you say about the index of refraction of oil relative to that of water?
7. In our discussion of thin-film interference, we looked at light *reflecting* from a thin film. Consider one light ray, the direct ray, that transmits through the film without reflecting. Consider a second ray, the reflected ray, that transmits through the first surface, reflects from the second, reflects again from the first, and then transmits out into the air, parallel to the direct ray. For normal incidence, how thick must the film be, in terms of the wavelength of light, for the outgoing rays to interfere destructively? Is it the same thickness as for reflected destructive interference?
8. Suppose that you are watching television connected to an antenna rather than a cable system. If an airplane flies near your location, you may notice wavering ghost images in the television picture. What might cause this?
9. If we are to observe interference in a thin film, why must the film not be very thick (on the order of a few wavelengths)?
10. A lens with outer radius of curvature R and index of re-

fraction n rests on a flat glass plate, and the combination is illuminated with white light from above. Is there a dark spot or a light spot at the center of the lens? What does it mean if the observed rings are noncircular?

11. Why is the lens on a high-quality camera coated with a thin film?

12. Why is it so much easier to perform interference experiments with a laser than with an ordinary light source?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 37.1 Conditions for Interference
Section 37.2 Young's Double-Slit Experiment

1. A laser beam (λ = 632.8 nm) is incident on two slits 0.200 mm apart. How far apart are the bright interference fringes on a screen 5.00 m away from the slits?

2. A Young's interference experiment is performed with monochromatic light. The separation between the slits is 0.500 mm, and the interference pattern on a screen 3.30 m away shows the first maximum 3.40 mm from the center of the pattern. What is the wavelength?

WEB **3.** Two radio antennas separated by 300 m as shown in Figure P37.3 simultaneously broadcast identical signals at the same wavelength. A radio in a car traveling due north receives the signals. (a) If the car is at the position of the second maximum, what is the wavelength of the signals? (b) How much farther must the car travel to encounter the next minimum in reception? (*Note:* Do not use the small-angle approximation in this problem.)

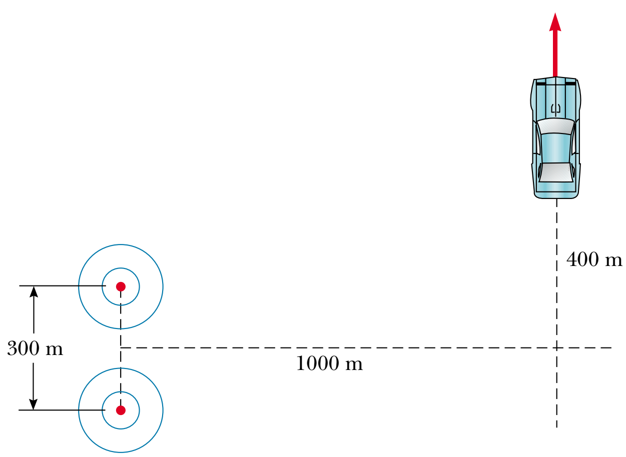


Figure P37.3

4. In a location where the speed of sound is 354 m/s, a 2 000-Hz sound wave impinges on two slits 30.0 cm apart. (a) At what angle is the first maximum located? (b) If the sound wave is replaced by 3.00-cm microwaves, what slit separation gives the same angle for the first maximum? (c) If the slit separation is 1.00 μm, what frequency of light gives the same first maximum angle?

WEB **5.** Young's double-slit experiment is performed with 589-nm light and a slit-to-screen distance of 2.00 m. The tenth interference minimum is observed 7.26 mm from the central maximum. Determine the spacing of the slits.

6. The two speakers of a boom box are 35.0 cm apart. A single oscillator makes the speakers vibrate in phase at a frequency of 2.00 kHz. At what angles, measured from the perpendicular bisector of the line joining the speakers, would a distant observer hear maximum sound intensity? minimum sound intensity? (Take the speed of sound as 340 m/s.)

7. A pair of narrow, parallel slits separated by 0.250 mm are illuminated by green light (λ = 546.1 nm). The interference pattern is observed on a screen 1.20 m away from the plane of the slits. Calculate the distance (a) from the central maximum to the first bright region on either side of the central maximum and (b) between the first and second dark bands.

8. Light with a wavelength of 442 nm passes through a double-slit system that has a slit separation d = 0.400 mm. Determine how far away a screen must be placed so that a dark fringe appears directly opposite both slits, with just one bright fringe between them.

9. A riverside warehouse has two open doors, as illustrated in Figure P37.9. Its walls are lined with sound-absorbing material. A boat on the river sounds its horn. To person A, the sound is loud and clear. To person B, the sound is barely audible. The principal wavelength of the sound waves is 3.00 m. Assuming that person B is at the position of the first minimum, determine the distance between the doors, center to center.

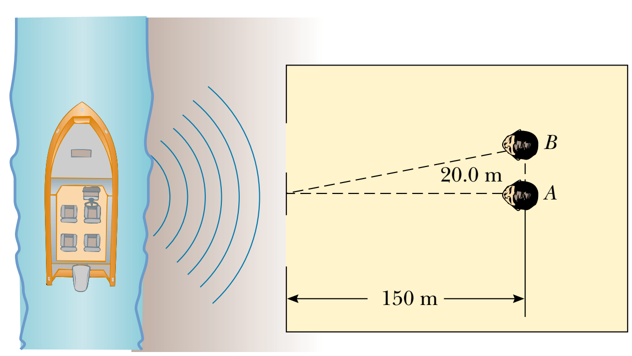


Figure P37.9

10. Two slits are separated by 0.320 mm. A beam of 500-nm light strikes the slits, producing an interference pattern. Determine the number of maxima observed in the angular range $-30.0° < \theta < 30.0°$.

11. In Figure 37.4 let $L = 1.20$ m and $d = 0.120$ mm, and assume that the slit system is illuminated with monochromatic 500-nm light. Calculate the phase difference between the two wavefronts arriving at point P when (a) $\theta = 0.500°$ and (b) $y = 5.00$ mm. (c) What is the value of θ for which the phase difference is 0.333 rad? (d) What is the value of θ for which the path difference is $\lambda/4$?

12. Coherent light rays of wavelength λ strike a pair of slits separated by distance d at an angle of θ_1, as shown in Figure P37.12. If an interference maximum is formed at an angle of θ_2 a great distance from the slits, show that $d(\sin\theta_2 - \sin\theta_1) = m\lambda$, where m is an integer.

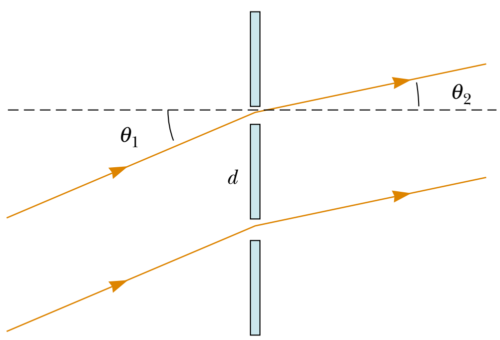


Figure P37.12

13. In the double-slit arrangement of Figure 37.4, $d = 0.150$ mm, $L = 140$ cm, $\lambda = 643$ nm, and $y = 1.80$ cm. (a) What is the path difference δ for the rays from the two slits arriving at point P? (b) Express this path difference in terms of λ. (c) Does point P correspond to a maximum, a minimum, or an intermediate condition?

Section 37.3 Intensity Distribution of the Double-Slit Interference Pattern

14. The intensity on the screen at a certain point in a double-slit interference pattern is 64.0% of the maximum value. (a) What minimum phase difference (in radians) between sources produces this result? (b) Express this phase difference as a path difference for 486.1-nm light.

WEB 15. In Figure 37.4, let $L = 120$ cm and $d = 0.250$ cm. The slits are illuminated with coherent 600-nm light. Calculate the distance y above the central maximum for which the average intensity on the screen is 75.0% of the maximum.

16. Two slits are separated by 0.180 mm. An interference pattern is formed on a screen 80.0 cm away by 656.3-nm light. Calculate the fraction of the maximum intensity 0.600 cm above the central maximum.

17. Two narrow parallel slits separated by 0.850 mm are illuminated by 600-nm light, and the viewing screen is 2.80 m away from the slits. (a) What is the phase difference between the two interfering waves on a screen at a point 2.50 mm from the central bright fringe? (b) What is the ratio of the intensity at this point to the intensity at the center of a bright fringe?

18. Monochromatic coherent light of amplitude E_0 and angular frequency ω passes through three parallel slits each separated by a distance d from its neighbor. (a) Show that the time-averaged intensity as a function of the angle θ is

$$I(\theta) = I_{\text{max}}\left[1 + 2\cos\left(\frac{2\pi d \sin\theta}{\lambda}\right)\right]^2$$

(b) Determine the ratio of the intensities of the primary and secondary maxima.

Section 37.4 Phasor Addition of Waves

19. Marie Cornu invented phasors in about 1880. This problem helps you to see their utility. Find the amplitude and phase constant of the sum of two waves represented by the expressions

$$E_1 = (12.0\text{ kN/C})\sin(15x - 4.5t)$$

and

$$E_2 = (12.0\text{ kN/C})\sin(15x - 4.5t + 70°)$$

(a) by using a trigonometric identity (see Appendix B) and (b) by representing the waves by phasors. (c) Find the amplitude and phase constant of the sum of the three waves represented by

$$E_1 = (12.0\text{ kN/C})\sin(15x - 4.5t + 70°)$$

$$E_2 = (15.5\text{ kN/C})\sin(15x - 4.5t - 80°)$$

and

$$E_3 = (17.0\text{ kN/C})\sin(15x - 4.5t + 160°)$$

20. The electric fields from three coherent sources are described by $E_1 = E_0\sin\omega t$, $E_2 = E_0\sin(\omega t + \phi)$, and $E_3 = E_0\sin(\omega t + 2\phi)$. Let the resultant field be represented by $E_P = E_R\sin(\omega t + \alpha)$. Use phasors to find E_R and α when (a) $\phi = 20.0°$, (b) $\phi = 60.0°$, and (c) $\phi = 120°$. (d) Repeat when $\phi = (3\pi/2)$ rad.

WEB 21. Determine the resultant of the two waves $E_1 = 6.0\sin(100\pi t)$ and $E_2 = 8.0\sin(100\pi t + \pi/2)$.

22. Suppose that the slit openings in a Young's double-slit experiment have different sizes so that the electric fields and the intensities from each slit are different. If $E_1 = E_{01}\sin(\omega t)$ and $E_2 = E_{02}\sin(\omega t + \phi)$, show that the resultant electric field is $E = E_0\sin(\omega t + \theta)$, where

$$E_0 = \sqrt{E_{01}^{\,2} + E_{02}^{\,2} + 2E_{01}E_{02}\cos\phi}$$

and

$$\sin\theta = \frac{E_{02}\sin\phi}{E_0}$$

23. Use phasors to find the resultant (magnitude and phase angle) of two fields represented by $E_1 = 12\sin\omega t$ and $E_2 = 18\sin(\omega t + 60°)$. (Note that in this case the amplitudes of the two fields are unequal.)

24. Two coherent waves are described by the expressions

$$E_1 = E_0 \sin\left(\frac{2\pi x_1}{\lambda} - 2\pi f t + \frac{\pi}{6}\right)$$

$$E_2 = E_0 \sin\left(\frac{2\pi x_2}{\lambda} - 2\pi f t + \frac{\pi}{8}\right)$$

Determine the relationship between x_1 and x_2 that produces constructive interference when the two waves are superposed.

25. When illuminated, four equally spaced parallel slits act as multiple coherent sources, each differing in phase from the adjacent one by an angle ϕ. Use a phasor diagram to determine the smallest value of ϕ for which the resultant of the four waves (assumed to be of equal amplitude) is zero.

26. Sketch a phasor diagram to illustrate the resultant of $E_1 = E_{01}\sin\omega t$ and $E_2 = E_{02}\sin(\omega t + \phi)$, where $E_{02} = 1.50E_{01}$ and $\pi/6 \le \phi \le \pi/3$. Use the sketch and the law of cosines to show that, for two coherent waves, the resultant intensity can be written in the form $I_R = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi$.

27. Consider N coherent sources described by $E_1 = E_0\sin(\omega t + \phi)$, $E_2 = E_0\sin(\omega t + 2\phi)$, $E_3 = E_0\sin(\omega t + 3\phi)$, . . . , $E_N = E_0\sin(\omega t + N\phi)$. Find the minimum value of ϕ for which $E_R = E_1 + E_2 + E_3 + \ldots + E_N$ is zero.

Section 37.5 Change of Phase Due to Reflection
Section 37.6 Interference in Thin Films

28. A soap bubble ($n = 1.33$) is floating in air. If the thickness of the bubble wall is 115 nm, what is the wavelength of the light that is most strongly reflected?

29. An oil film ($n = 1.45$) floating on water is illuminated by white light at normal incidence. The film is 280 nm thick. Find (a) the dominant observed color in the reflected light and (b) the dominant color in the transmitted light. Explain your reasoning.

30. A thin film of oil ($n = 1.25$) is located on a smooth, wet pavement. When viewed perpendicular to the pavement, the film appears to be predominantly red (640 nm) and has no blue color (512 nm). How thick is the oil film?

31. A possible means for making an airplane invisible to radar is to coat the plane with an antireflective polymer. If radar waves have a wavelength of 3.00 cm and the index of refraction of the polymer is $n = 1.50$, how thick would you make the coating?

32. A material having an index of refraction of 1.30 is used to coat a piece of glass ($n = 1.50$). What should be the minimum thickness of this film if it is to minimize reflection of 500-nm light?

33. A film of MgF_2 ($n = 1.38$) having a thickness of 1.00×10^{-5} cm is used to coat a camera lens. Are any wavelengths in the visible spectrum intensified in the reflected light?

34. Astronomers observe the chromosphere of the Sun with a filter that passes the red hydrogen spectral line of wavelength 656.3 nm, called the H_α line. The filter consists of a transparent dielectric of thickness d held between two partially aluminized glass plates. The filter is held at a constant temperature. (a) Find the minimum value of d that produces maximum transmission of perpendicular H_α light, if the dielectric has an index of refraction of 1.378. (b) Assume that the temperature of the filter increases above its normal value and that its index of refraction does not change significantly. What happens to the transmitted wavelength? (c) The dielectric will also pass what near-visible wavelength? One of the glass plates is colored red to absorb this light.

35. A beam of 580-nm light passes through two closely spaced glass plates, as shown in Figure P37.35. For what minimum nonzero value of the plate separation d is the transmitted light bright?

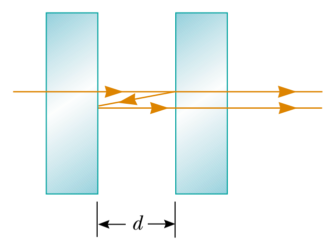


Figure P37.35

36. When a liquid is introduced into the air space between the lens and the plate in a Newton's-rings apparatus, the diameter of the tenth ring changes from 1.50 to 1.31 cm. Find the index of refraction of the liquid.

WEB **37.** An air wedge is formed between two glass plates separated at one edge by a very fine wire, as shown in Figure P37.37. When the wedge is illuminated from above by 600-nm light, 30 dark fringes are observed. Calculate the radius of the wire.

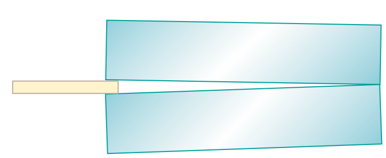


Figure P37.37 Problems 37 and 38.

38. Two rectangular flat glass plates ($n = 1.52$) are in contact along one end and separated along the other end by a sheet of paper 4.00×10^{-3} cm thick (see Fig. P37.37). The top plate is illuminated by monochromatic light ($\lambda = 546.1$ nm). Calculate the number of dark parallel bands crossing the top plate (include the dark band at zero thickness along the edge of contact between the two plates).

39. Two glass plates 10.0 cm long are in contact at one end and separated at the other end by a thread 0.050 0 mm in diameter. Light containing the two wavelengths 400 nm and 600 nm is incident perpendicularly. At what distance from the contact point is the next dark fringe?

(Optional)
Section 37.7 The Michelson Interferometer

40. Light of wavelength 550.5 nm is used to calibrate a Michelson interferometer, and mirror M_1 is moved 0.180 mm. How many dark fringes are counted?

41. Mirror M_1 in Figure 37.22 is displaced a distance ΔL. During this displacement, 250 fringe reversals (formation of successive dark or bright bands) are counted. The light being used has a wavelength of 632.8 nm. Calculate the displacement ΔL.

42. Monochromatic light is beamed into a Michelson interferometer. The movable mirror is displaced 0.382 mm; this causes the interferometer pattern to reproduce itself 1 700 times. Determine the wavelength and the color of the light.

43. One leg of a Michelson interferometer contains an evacuated cylinder 3.00 cm long having glass plates on each end. A gas is slowly leaked into the cylinder until a pressure of 1 atm is reached. If 35 bright fringes pass on the screen when light of wavelength 633 nm is used, what is the index of refraction of the gas?

44. One leg of a Michelson interferometer contains an evacuated cylinder of length L having glass plates on each end. A gas is slowly leaked into the cylinder until a pressure of 1 atm is reached. If N bright fringes pass on the screen when light of wavelength λ is used, what is the index of refraction of the gas?

ADDITIONAL PROBLEMS

45. One radio transmitter A operating at 60.0 MHz is 10.0 m from another similar transmitter B that is 180° out of phase with transmitter A. How far must an observer move from transmitter A toward transmitter B along the line connecting A and B to reach the nearest point where the two beams are in phase?

46. Raise your hand and hold it flat. Think of the space between your index finger and your middle finger as one slit, and think of the space between middle finger and ring finger as a second slit. (a) Consider the interference resulting from sending coherent visible light perpendicularly through this pair of openings. Compute an order-of-magnitude estimate for the angle between adjacent zones of constructive interference. (b) To make the angles in the interference pattern easy to measure with a plastic protractor, you should use an electromagnetic wave with frequency of what order of magnitude? How is this wave classified on the electromagnetic spectrum?

47. In a Young's double-slit experiment using light of wavelength λ, a thin piece of Plexiglas having index of refraction n covers one of the slits. If the center point on the screen is a dark spot instead of a bright spot, what is the minimum thickness of the Plexiglas?

48. **Review Problem.** A flat piece of glass is held stationary and horizontal above the flat top end of a 10.0-cm-long vertical metal rod that has its lower end rigidly fixed. The thin film of air between the rod and glass is observed to be bright by reflected light when it is illuminated by light of wavelength 500 nm. As the temperature is slowly increased by 25.0°C, the film changes from bright to dark and back to bright 200 times. What is the coefficient of linear expansion of the metal?

49. A certain crude oil has an index of refraction of 1.25. A ship dumps 1.00 m^3 of this oil into the ocean, and the oil spreads into a thin uniform slick. If the film produces a first-order maximum of light of wavelength 500 nm normally incident on it, how much surface area of the ocean does the oil slick cover? Assume that the index of refraction of the ocean water is 1.34.

50. Interference effects are produced at point P on a screen as a result of direct rays from a 500-nm source and reflected rays off the mirror, as shown in Figure P37.50. If the source is 100 m to the left of the screen and 1.00 cm above the mirror, find the distance y (in millimeters) to the first dark band above the mirror.

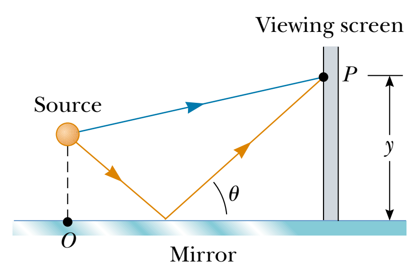


Figure P37.50

51. Astronomers observed a 60.0-MHz radio source both directly and by reflection from the sea. If the receiving dish is 20.0 m above sea level, what is the angle of the radio source above the horizon at first maximum?

52. The waves from a radio station can reach a home receiver by two paths. One is a straight-line path from transmitter to home, a distance of 30.0 km. The second path is by reflection from the ionosphere (a layer of ionized air molecules high in the atmosphere). Assume that this reflection takes place at a point midway between the receiver and the transmitter. The wavelength broadcast by the radio station is 350 m. Find the minimum height of the ionospheric layer that produces destructive inter-

ference between the direct and reflected beams. (Assume that no phase changes occur on reflection.)

53. Measurements are made of the intensity distribution in a Young's interference pattern (see Fig. 37.6). At a particular value of y, it is found that $I/I_{max} = 0.810$ when 600-nm light is used. What wavelength of light should be used if the relative intensity at the same location is to be reduced to 64.0%?

54. In a Young's interference experiment, the two slits are separated by 0.150 mm, and the incident light includes light of wavelengths $\lambda_1 = 540$ nm and $\lambda_2 = 450$ nm. The overlapping interference patterns are formed on a screen 1.40 m from the slits. Calculate the minimum distance from the center of the screen to the point where a bright line of the λ_1 light coincides with a bright line of the λ_2 light.

55. An air wedge is formed between two glass plates in contact along one edge and slightly separated at the opposite edge. When the plates are illuminated with monochromatic light from above, the reflected light has 85 dark fringes. Calculate the number of dark fringes that would appear if water ($n = 1.33$) were to replace the air between the plates.

56. Our discussion of the techniques for determining constructive and destructive interference by reflection from a thin film in air has been confined to rays striking the film at nearly normal incidence. Assume that a ray is incident at an angle of 30.0° (relative to the normal) on a film with an index of refraction of 1.38. Calculate the minimum thickness for constructive interference if the light is sodium light with a wavelength of 590 nm.

57. The condition for constructive interference by reflection from a thin film in air as developed in Section 37.6 assumes nearly normal incidence. Show that if the light is incident on the film at a nonzero angle ϕ_1 (relative to the normal), then the condition for constructive interference is $2nt \cos \theta_2 = (m + \frac{1}{2})\lambda$, where θ_2 is the angle of refraction.

58. (a) Both sides of a uniform film that has index of refraction n and thickness d are in contact with air. For normal incidence of light, an intensity minimum is observed in the reflected light at λ_2, and an intensity maximum is observed at λ_1, where $\lambda_1 > \lambda_2$. If no intensity minima are observed between λ_1 and λ_2, show that the integer m in Equations 37.16 and 37.17 is given by $m = \lambda_1/2(\lambda_1 - \lambda_2)$. (b) Determine the thickness of the film if $n = 1.40$, $\lambda_1 = 500$ nm, and $\lambda_2 = 370$ nm.

59. Figure P37.59 shows a radio wave transmitter and a receiver separated by a distance d and located a distance h above the ground. The receiver can receive signals both directly from the transmitter and indirectly from signals that reflect off the ground. Assume that the ground is level between the transmitter and receiver and that a 180° phase shift occurs upon reflection. Determine the longest wavelengths that interfere (a) constructively and (b) destructively.

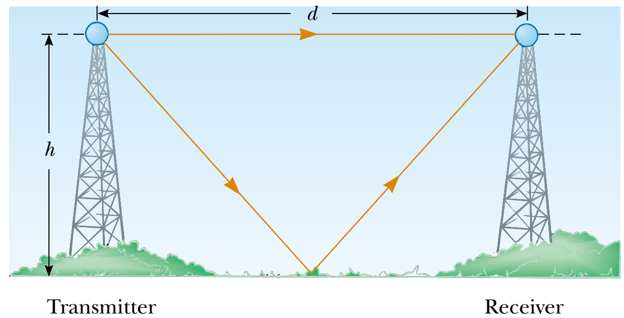


Figure P37.59

60. Consider the double-slit arrangement shown in Figure P37.60, where the separation d is 0.300 mm and the distance L is 1.00 m. A sheet of transparent plastic ($n = 1.50$) 0.050 0 mm thick (about the thickness of this page) is placed over the upper slit. As a result, the central maximum of the interference pattern moves upward a distance y'. Find y'.

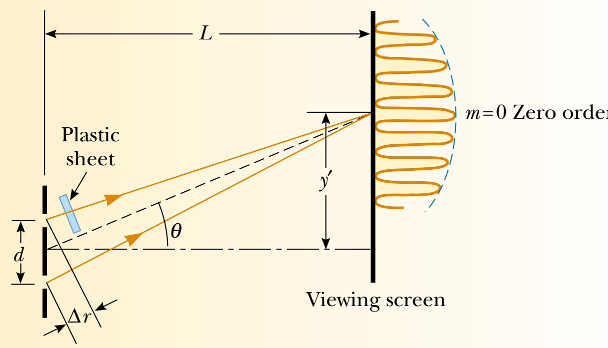


Figure P37.60 Problems 60 and 61.

61. Consider the double-slit arrangement shown in Figure P37.60, where the slit separation is d and the slit to screen distance is L. A sheet of transparent plastic having an index of refraction n and thickness t is placed over the upper slit. As a result, the central maximum of the interference pattern moves upward a distance y'. Find y'.

62. Waves broadcast by a 1 500-kHz radio station arrive at a home receiver by two paths. One is a direct path, and the other is from reflection off an airplane directly above the receiver. The airplane is approximately 100 m above the receiver, and the direct distance from station to home is 20.0 km. What is the precise height of the airplane if destructive interference is occurring? (Assume that no phase change occurs on reflection.)

63. In a Newton's-rings experiment, a plano-convex glass ($n = 1.52$) lens having a diameter of 10.0 cm is placed on a flat plate, as shown in Figure 37.18a. When 650-nm light is incident normally, 55 bright rings are observed, with the last ring right on the edge of the lens. (a) What is the radius of curvature of the convex surface of the lens? (b) What is the focal length of the lens?

64. A piece of transparent material having an index of re-

fraction n is cut into the shape of a wedge, as shown in Figure P37.64. The angle of the wedge is small, and monochromatic light of wavelength λ is normally incident from above. If the height of the wedge is h and the width is ℓ, show that bright fringes occur at the positions $x = \lambda\ell(m + \frac{1}{2})/2hn$ and that dark fringes occur at the positions $x = \lambda\ell m/2hn$, where $m = 0, 1, 2, \ldots$ and x is measured as shown.

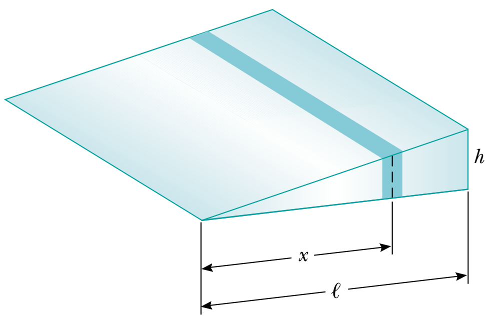


Figure P37.64

65. Use phasor addition to find the resultant amplitude and phase constant when the following three harmonic functions are combined: $E_1 = \sin(\omega t + \pi/6)$, $E_2 = 3.0\sin(\omega t + 7\pi/2)$, $E_3 = 6.0\sin(\omega t + 4\pi/3)$.

66. A plano-convex lens having a radius of curvature of $r = 4.00$ m is placed on a concave reflecting surface whose radius of curvature is $R = 12.0$ m, as shown in Figure P37.66. Determine the radius of the 100th bright ring if 500-nm light is incident normal to the flat surface of the lens.

67. A plano-convex lens has index of refraction n. The curved side of the lens has radius of curvature R and rests on a flat glass surface of the same index of refraction, with a film of index n_{film} between them. The lens is illuminated from above by light of wavelength λ. Show that the dark Newton's rings have radii given approximately by

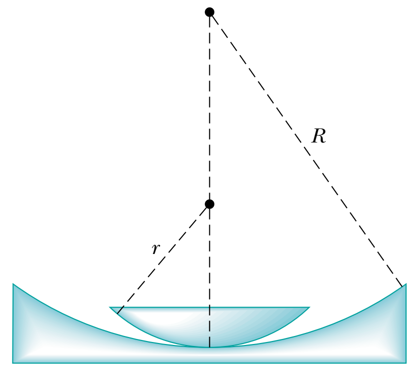


Figure P37.66

$$r \cong \sqrt{m\lambda R/n_{\text{film}}}$$

where m is an integer and r is much less than R.

68. A soap film ($n = 1.33$) is contained within a rectangular wire frame. The frame is held vertically so that the film drains downward and becomes thicker at the bottom than at the top, where the thickness is essentially zero. The film is viewed in white light with near-normal incidence, and the first violet ($\lambda = 420$ nm) interference band is observed 3.00 cm from the top edge of the film. (a) Locate the first red ($\lambda = 680$ nm) interference band. (b) Determine the film thickness at the positions of the violet and red bands. (c) What is the wedge angle of the film?

69. Interference fringes are produced using Lloyd's mirror and a 606-nm source, as shown in Figure 37.14. Fringes 1.20 mm apart are formed on a screen 2.00 m from the real source S. Find the vertical distance h of the source above the reflecting surface.

70. Slit 1 of a double slit is wider than slit 2, so that the light from slit 1 has an amplitude 3.00 times that of the light from slit 2. Show that Equation 37.11 is replaced by the equation $I = (4I_{\max}/9)(1 + 3\cos^2\phi/2)$ for this situation.

Answers to Quick Quizzes

37.1 Bands of light along the orange lines interspersed with dark bands running along the dashed black lines.

37.2 At location B. At A, which is on a line of constructive interference, the water surface undulates so much that you probably could not determine the depth. Because B is on a line of destructive interference, the water level does not change, and you should be able to read the ruler easily.

37.3 The graph is shown in Figure QQA37.1. The width of the primary maxima is slightly narrower than the $N = 5$ primary width but wider than the $N = 10$ primary width. Because $N = 6$, the secondary maxima are $\frac{1}{36}$ as intense as the primary maxima.

37.4 The greater the variation in thickness, the narrower the bands of color (like the lines on a topographic map). The widest bands are the gold ones along the left edge of the photograph and at the bottom right corner of the razor blade. Thus, the thickness of the oil film changes most slowly with position in these areas.

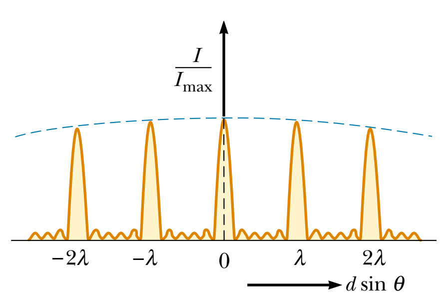


Figure QQA37.1

PUZZLER

At sunset, the sky is ablaze with brilliant reds, pinks, and oranges. Yet, we wouldn't be able to see this sunset were it not for the fact that someone else is simultaneously seeing a blue sky. What causes the beautiful colors of a sunset, and why must the sky be blue somewhere else for us to enjoy one? *(© W. A. Banaszewski/Visuals Unlimited)*

chapter 38

Diffraction and Polarization

Chapter Outline

38.1 Introduction to Diffraction

38.2 Diffraction from Narrow Slits

38.3 Resolution of Single-Slit and Circular Apertures

38.4 The Diffraction Grating

38.5 *(Optional)* Diffraction of X-Rays by Crystals

38.6 Polarization of Light Waves

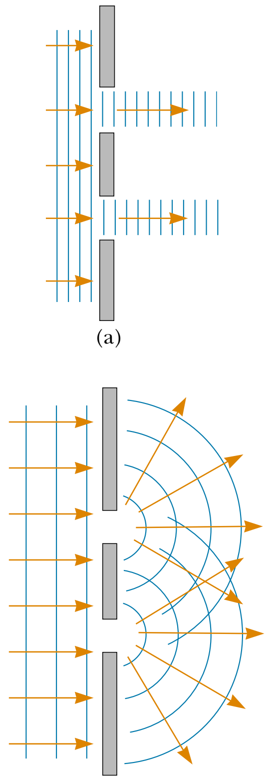


Figure 38.1 (a) If light waves did not spread out after passing through the slits, no interference would occur. (b) The light waves from the two slits overlap as they spread out, filling what we expect to be shadowed regions with light and producing interference fringes.

When light waves pass through a small aperture, an interference pattern is observed rather than a sharp spot of light. This behavior indicates that light, once it has passed through the aperture, spreads beyond the narrow path defined by the aperture into regions that would be in shadow if light traveled in straight lines. Other waves, such as sound waves and water waves, also have this property of spreading when passing through apertures or by sharp edges. This phenomenon, known as diffraction, can be described only with a wave model for light.

In Chapter 34, we learned that electromagnetic waves are transverse. That is, the electric and magnetic field vectors are perpendicular to the direction of wave propagation. In this chapter, we see that under certain conditions these transverse waves can be polarized in various ways.

38.1 INTRODUCTION TO DIFFRACTION

In Section 37.2 we learned that an interference pattern is observed on a viewing screen when two slits are illuminated by a single-wavelength light source. If the light traveled only in its original direction after passing through the slits, as shown in Figure 38.1a, the waves would not overlap and no interference pattern would be seen. Instead, Huygens's principle requires that the waves spread out from the slits as shown in Figure 38.1b. In other words, the light deviates from a straight-line path and enters the region that would otherwise be shadowed. As noted in Section 35.1, this divergence of light from its initial line of travel is called **diffraction.**

In general, diffraction occurs when waves pass through small openings, around obstacles, or past sharp edges, as shown in Figure 38.2. When an opaque object is placed between a point source of light and a screen, no sharp boundary exists on the screen between a shadowed region and an illuminated region. The illuminated region above the shadow of the object contains alternating light and dark fringes. Such a display is called a **diffraction pattern.**

Figure 38.3 shows a diffraction pattern associated with the shadow of a penny. A bright spot occurs at the center, and circular fringes extend outward from the shadow's edge. We can explain the central bright spot only by using the wave the-

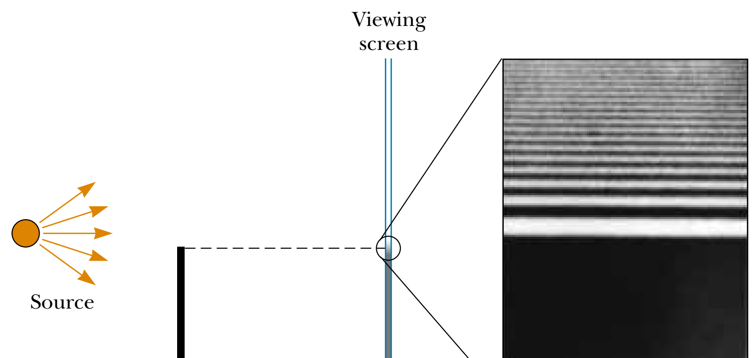


Figure 38.2 Light from a small source passes by the edge of an opaque object. We might expect no light to appear on the screen below the position of the edge of the object. In reality, light bends around the top edge of the object and enters this region. Because of these effects, a diffraction pattern consisting of bright and dark fringes appears in the region above the edge of the object.

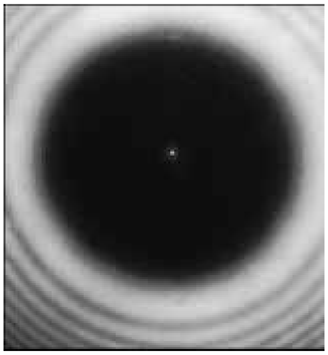


Figure 38.3 Diffraction pattern created by the illumination of a penny, with the penny positioned midway between screen and light source.

ory of light, which predicts constructive interference at this point. From the viewpoint of geometric optics (in which light is viewed as rays traveling in straight lines), we expect the center of the shadow to be dark because that part of the viewing screen is completely shielded by the penny.

It is interesting to point out an historical incident that occurred shortly before the central bright spot was first observed. One of the supporters of geometric optics, Simeon Poisson, argued that if Augustin Fresnel's wave theory of light were valid, then a central bright spot should be observed in the shadow of a circular object illuminated by a point source of light. To Poisson's astonishment, the spot was observed by Dominique Arago shortly thereafter. Thus, Poisson's prediction reinforced the wave theory rather than disproving it.

In this chapter we restrict our attention to **Fraunhofer diffraction,** which occurs, for example, when all the rays passing through a narrow slit are approximately parallel to one another. This can be achieved experimentally either by placing the screen far from the opening used to create the diffraction or by using a converging lens to focus the rays once they pass through the opening, as shown in Figure 38.4a. A bright fringe is observed along the axis at $\theta = 0$, with alternating dark and bright fringes occurring on either side of the central bright one. Figure 38.4b is a photograph of a single-slit Fraunhofer diffraction pattern.

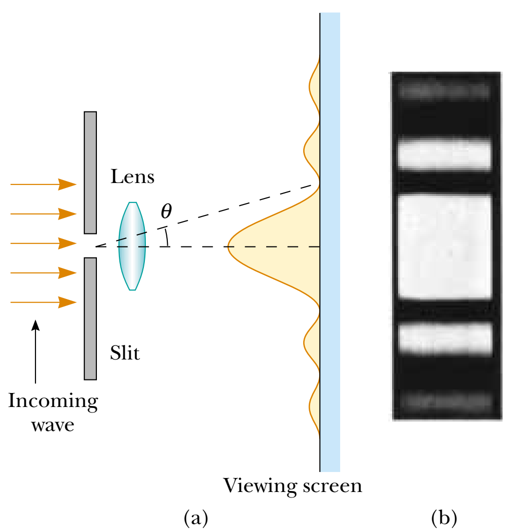


Figure 38.4 (a) Fraunhofer diffraction pattern of a single slit. The pattern consists of a central bright fringe flanked by much weaker maxima alternating with dark fringes (drawing not to scale). (b) Photograph of a single-slit Fraunhofer diffraction pattern.

38.2 DIFFRACTION FROM NARROW SLITS

Until now, we have assumed that slits are point sources of light. In this section, we abandon that assumption and see how the finite width of slits is the basis for understanding Fraunhofer diffraction.

We can deduce some important features of this phenomenon by examining waves coming from various portions of the slit, as shown in Figure 38.5. According to Huygens's principle, **each portion of the slit acts as a source of light waves.** Hence, light from one portion of the slit can interfere with light from another portion, and the resultant light intensity on a viewing screen depends on the direction θ.

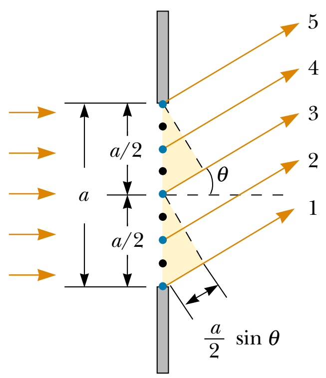


Figure 38.5 Diffraction of light by a narrow slit of width a. Each portion of the slit acts as a point source of light waves. The path difference between rays 1 and 3 or between rays 2 and 4 is $(a/2)\sin\theta$ (drawing not to scale).

To analyze the diffraction pattern, it is convenient to divide the slit into two halves, as shown in Figure 38.5. Keeping in mind that all the waves are in phase as they leave the slit, consider rays 1 and 3. As these two rays travel toward a viewing screen far to the right of the figure, ray 1 travels farther than ray 3 by an amount equal to the path difference $(a/2)\sin\theta$, where a is the width of the slit. Similarly, the path difference between rays 2 and 4 is also $(a/2)\sin\theta$. If this path difference is exactly half a wavelength (corresponding to a phase difference of 180°), then the two waves cancel each other and destructive interference results. This is true for any two rays that originate at points separated by half the slit width because the phase difference between two such points is 180°. Therefore, waves from the upper half of the slit interfere destructively with waves from the lower half when

$$\frac{a}{2}\sin\theta = \frac{\lambda}{2}$$

or when

$$\sin\theta = \frac{\lambda}{a}$$

If we divide the slit into four equal parts and use similar reasoning, we find that the viewing screen is also dark when

$$\sin\theta = \frac{2\lambda}{a}$$

Likewise, we can divide the slit into six equal parts and show that darkness occurs on the screen when

$$\sin\theta = \frac{3\lambda}{a}$$

Therefore, the general condition for destructive interference is

Condition for destructive interference

$$\boxed{\sin\theta = m\frac{\lambda}{a}} \qquad m = \pm 1, \pm 2, \pm 3, \ . \ . \ . \qquad \textbf{(38.1)}$$

This equation gives the values of θ for which the diffraction pattern has zero light intensity—that is, when a dark fringe is formed. However, it tells us nothing about the variation in light intensity along the screen. The general features of the intensity distribution are shown in Figure 38.6. A broad central bright fringe is ob-

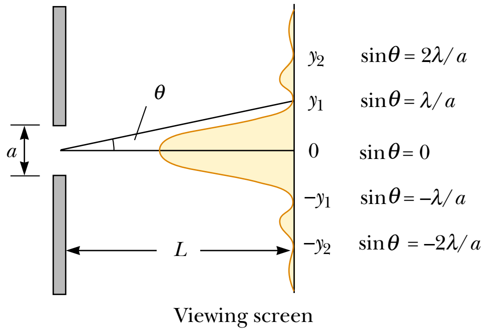
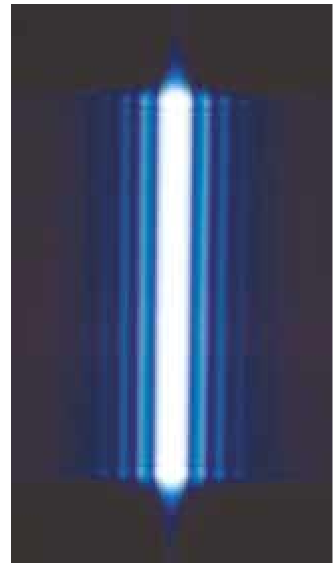


Figure 38.6 Intensity distribution for a Fraunhofer diffraction pattern from a single slit of width a. The positions of two minima on each side of the central maximum are labeled (drawing not to scale).

served; this fringe is flanked by much weaker bright fringes alternating with dark fringes. The various dark fringes occur at the values of θ that satisfy Equation 38.1. Each bright-fringe peak lies approximately halfway between its bordering dark-fringe minima. Note that the central bright maximum is twice as wide as the secondary maxima.



The diffraction pattern that appears on a screen when light passes through a narrow vertical slit. The pattern consists of a broad central bright fringe and a series of less intense and narrower side bright fringes.

Quick Quiz 38.1

If the door to an adjoining room is slightly ajar, why is it that you can hear sounds from the room but cannot see much of what is happening in the room?

EXAMPLE 38.1 Where Are the Dark Fringes?

Light of wavelength 580 nm is incident on a slit having a width of 0.300 mm. The viewing screen is 2.00 m from the slit. Find the positions of the first dark fringes and the width of the central bright fringe.

Solution The two dark fringes that flank the central bright fringe correspond to $m = \pm 1$ in Equation 38.1. Hence, we find that

$$\sin\theta = \pm\frac{\lambda}{a} = \pm\frac{5.80 \times 10^{-7}\ \text{m}}{0.300 \times 10^{-3}\ \text{m}} = \pm 1.93 \times 10^{-3}$$

From the triangle in Figure 38.6, note that $\tan\theta = y_1/L$. Because θ is very small, we can use the approximation $\sin\theta \approx \tan\theta$; thus, $\sin\theta \approx y_1/L$. Therefore, the positions of the first minima measured from the central axis are given by

$$y_1 \approx L\sin\theta = \pm L\frac{\lambda}{a} = \boxed{\pm 3.87 \times 10^{-3}\ \text{m}}$$

The positive and negative signs correspond to the dark fringes on either side of the central bright fringe. Hence, the width of the central bright fringe is equal to $2|y_1| = 7.74 \times 10^{-3}\ \text{m} = 7.74\ \text{mm}$. Note that this value is much greater than the width of the slit. However, as the slit width is increased, the diffraction pattern narrows, corresponding to smaller values of θ. In fact, for large values of a, the various maxima and minima are so closely spaced that only a large central bright area resembling the geometric image of the slit is observed. This is of great importance in the design of lenses used in telescopes, microscopes, and other optical instruments.

Exercise Determine the width of the first-order ($m = 1$) bright fringe.

Answer 3.87 mm.

Intensity of Single-Slit Diffraction Patterns

We can use phasors to determine the light intensity distribution for a single-slit diffraction pattern. Imagine a slit divided into a large number of small zones, each of width Δy as shown in Figure 38.7. Each zone acts as a source of coherent radiation,

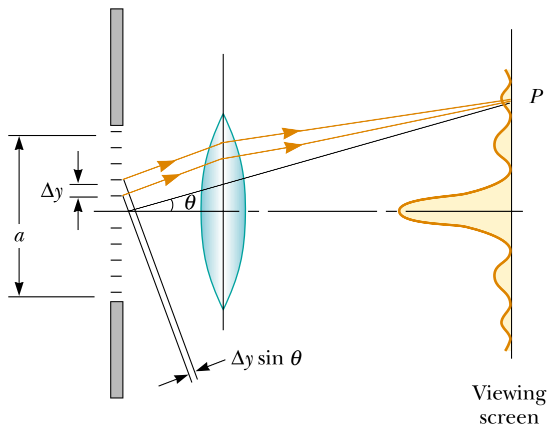


Figure 38.7 Fraunhofer diffraction by a single slit. The light intensity at point P is the resultant of all the incremental electric field magnitudes from zones of width Δy.

QuickLab

Make a V with your index and middle fingers. Hold your hand up very close to your eye so that you are looking between your two fingers toward a bright area. Now bring the fingers together until there is only a very tiny slit between them. You should be able to see a series of parallel lines. Although the lines appear to be located in the narrow space between your fingers, what you are actually seeing is a diffraction pattern cast upon your retina.

and each contributes an incremental electric field of magnitude ΔE at some point P on the screen. We obtain the total electric field magnitude E at point P by summing the contributions from all the zones. The light intensity at point P is proportional to the square of the magnitude of the electric field (see Section 37.3).

The incremental electric field magnitudes between adjacent zones are out of phase with one another by an amount $\Delta\beta$, where the phase difference $\Delta\beta$ is related to the path difference $\Delta y \sin\theta$ between adjacent zones by the expression

$$\Delta\beta = \frac{2\pi}{\lambda}\,\Delta y \sin\theta \qquad \textbf{(38.2)}$$

To find the magnitude of the total electric field on the screen at any angle θ, we sum the incremental magnitudes ΔE due to each zone. For small values of θ, we can assume that all the ΔE values are the same. It is convenient to use phasor diagrams for various angles, as shown in Figure 38.8. When $\theta = 0$, all phasors are aligned as shown in Figure 38.8a because all the waves from the various zones are in phase. In this case, the total electric field at the center of the screen is $E_0 = N\,\Delta E$, where N is the number of zones. The resultant magnitude E_R at some small angle θ is shown in Figure 38.8b, where each phasor differs in phase from an adjacent one by an amount $\Delta\beta$. In this case, E_R is the vector sum of the incremental

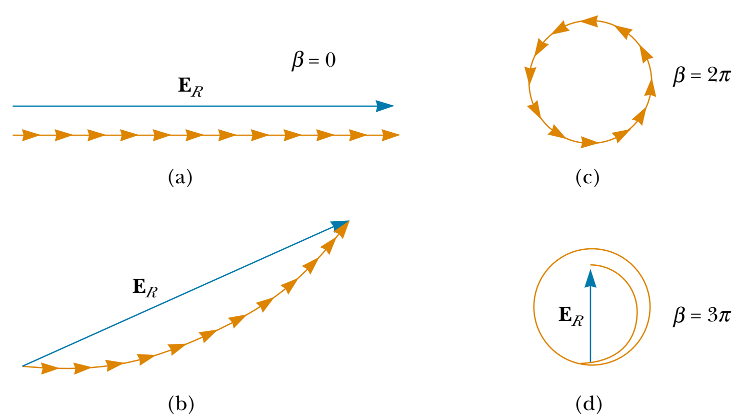


Figure 38.8 Phasor diagrams for obtaining the various maxima and minima of a single-slit diffraction pattern.

magnitudes and hence is given by the length of the chord. Therefore, $E_R < E_0$. The total phase difference β between waves from the top and bottom portions of the slit is

$$\beta = N\Delta\beta = \frac{2\pi}{\lambda} N\Delta y \sin\theta = \frac{2\pi}{\lambda} a \sin\theta \qquad \textbf{(38.3)}$$

where $a = N\Delta y$ is the width of the slit.

As θ increases, the chain of phasors eventually forms the closed path shown in Figure 38.8c. At this point, the vector sum is zero, and so $E_R = 0$, corresponding to the first minimum on the screen. Noting that $\beta = N\Delta\beta = 2\pi$ in this situation, we see from Equation 38.3 that

$$2\pi = \frac{2\pi}{\lambda} a \sin\theta$$

$$\sin\theta = \frac{\lambda}{a}$$

That is, the first minimum in the diffraction pattern occurs where $\sin\theta = \lambda/a$; this is in agreement with Equation 38.1.

At greater values of θ, the spiral chain of phasors tightens. For example, Figure 38.8d represents the situation corresponding to the second maximum, which occurs when $\beta = 360° + 180° = 540°$ (3π rad). The second minimum (two complete circles, not shown) corresponds to $\beta = 720°$ (4π rad), which satisfies the condition $\sin\theta = 2\lambda/a$.

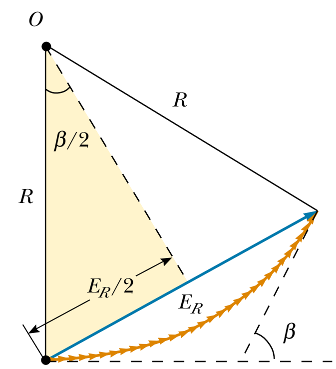


Figure 38.9 Phasor diagram for a large number of coherent sources. All the ends of the phasors lie on the circular red arc of radius R. The resultant electric field magnitude E_R equals the length of the chord.

We can obtain the total electric field magnitude E_R and light intensity I at any point P on the screen in Figure 38.7 by considering the limiting case in which Δy becomes infinitesimal (dy) and N approaches ∞. In this limit, the phasor chains in Figure 38.8 become the red curve of Figure 38.9. The arc length of the curve is E_0 because it is the sum of the magnitudes of the phasors (which is the total electric field magnitude at the center of the screen). From this figure, we see that at some angle θ, the resultant electric field magnitude E_R on the screen is equal to the chord length. From the triangle containing the angle $\beta/2$, we see that

$$\sin\frac{\beta}{2} = \frac{E_R/2}{R}$$

where R is the radius of curvature. But the arc length E_0 is equal to the product $R\beta$, where β is measured in radians. Combining this information with the previous expression gives

$$E_R = 2R\sin\frac{\beta}{2} = 2\left(\frac{E_0}{\beta}\right)\sin\frac{\beta}{2} = E_0\left[\frac{\sin(\beta/2)}{\beta/2}\right]$$

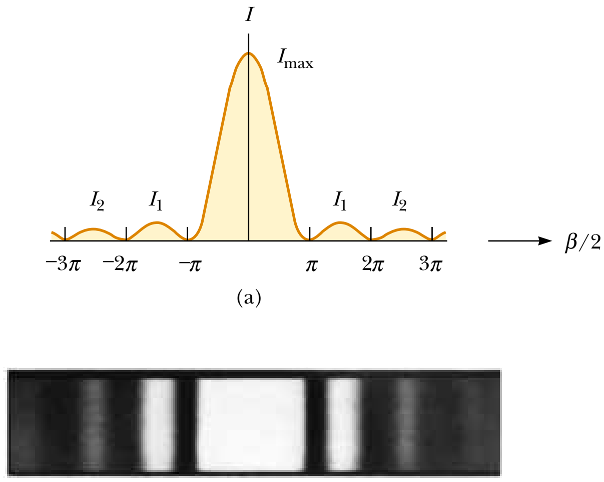
Because the resultant light intensity I at point P on the screen is proportional to the square of the magnitude E_R, we find that

$$I = I_{\max}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 \qquad \textbf{(38.4)}$$

Intensity of a single-slit Fraunhofer diffraction pattern

where $I_{\max}$ is the intensity at $\theta = 0$ (the central maximum). Substituting the expression for β (Eq. 38.3) into Equation 38.4, we have

$$I = I_{\max}\left[\frac{\sin(\pi a\sin\theta/\lambda)}{\pi a\sin\theta/\lambda}\right]^2 \qquad \textbf{(38.5)}$$



(a)

(b)

Figure 38.10 (a) A plot of light intensity I versus $\beta/2$ for the single-slit Fraunhofer diffraction pattern. (b) Photograph of a single-slit Fraunhofer diffraction pattern.

From this result, we see that minima occur when

$$\frac{\pi a \sin\theta}{\lambda} = m\pi$$

or

Condition for intensity minima

$$\sin\theta = m\frac{\lambda}{a} \qquad m = \pm 1, \pm 2, \pm 3, \ldots$$

in agreement with Equation 38.1.

Figure 38.10a represents a plot of Equation 38.5, and Figure 38.10b is a photograph of a single-slit Fraunhofer diffraction pattern. Note that most of the light intensity is concentrated in the central bright fringe.

EXAMPLE 38.2 Relative Intensities of the Maxima

Find the ratio of the intensities of the secondary maxima to the intensity of the central maximum for the single-slit Fraunhofer diffraction pattern.

Solution To a good approximation, the secondary maxima lie midway between the zero points. From Figure 38.10a, we see that this corresponds to $\beta/2$ values of $3\pi/2$, $5\pi/2$, $7\pi/2$, ... Substituting these values into Equation 38.4 gives for the first two ratios

$$\frac{I_1}{I_{\max}} = \left[\frac{\sin(3\pi/2)}{(3\pi/2)}\right]^2 = \frac{1}{9\pi^2/4} = \boxed{0.045}$$

$$\frac{I_2}{I_{\max}} = \left[\frac{\sin(5\pi/2)}{5\pi/2}\right]^2 = \frac{1}{25\pi^2/4} = \boxed{0.016}$$

That is, the first secondary maxima (the ones adjacent to the central maximum) have an intensity of 4.5% that of the central maximum, and the next secondary maxima have an intensity of 1.6% that of the central maximum.

Exercise Determine the intensity, relative to the central maximum, of the secondary maxima corresponding to $m = \pm 3$.

Answer 0.008 3.

Intensity of Two-Slit Diffraction Patterns

When more than one slit is present, we must consider not only diffraction due to the individual slits but also the interference of the waves coming from different slits. You may have noticed the curved dashed line in Figure 37.13, which indicates a decrease in intensity of the interference maxima as θ increases. This decrease is

due to diffraction. To determine the effects of both interference and diffraction, we simply combine Equation 37.12 and Equation 38.5:

$$I = I_{\text{max}} \cos^2\left(\frac{\pi d \sin\theta}{\lambda}\right)\left[\frac{\sin(\pi a \sin\theta/\lambda)}{\pi a \sin\theta/\lambda}\right]^2 \qquad \textbf{(38.6)}$$

Although this formula looks complicated, it merely represents the diffraction pattern (the factor in brackets) acting as an "envelope" for a two-slit interference pattern (the cosine-squared factor), as shown in Figure 38.11.

Equation 37.2 indicates the conditions for interference maxima as $d \sin\theta = m\lambda$, where d is the distance between the two slits. Equation 38.1 specifies that the first diffraction minimum occurs when $a \sin\theta = \lambda$, where a is the slit width. Dividing Equation 37.2 by Equation 38.1 (with $m = 1$) allows us to determine which interference maximum coincides with the first diffraction minimum:

$$\frac{d \sin\theta}{a \sin\theta} = \frac{m\lambda}{\lambda}$$

$$\frac{d}{a} = m \qquad \textbf{(38.7)}$$

In Figure 38.11, $d/a = 18\ \mu\text{m}/3.0\ \mu\text{m} = 6$. Thus, the sixth interference maximum (if we count the central maximum as $m = 0$) is aligned with the first diffraction minimum and cannot be seen.

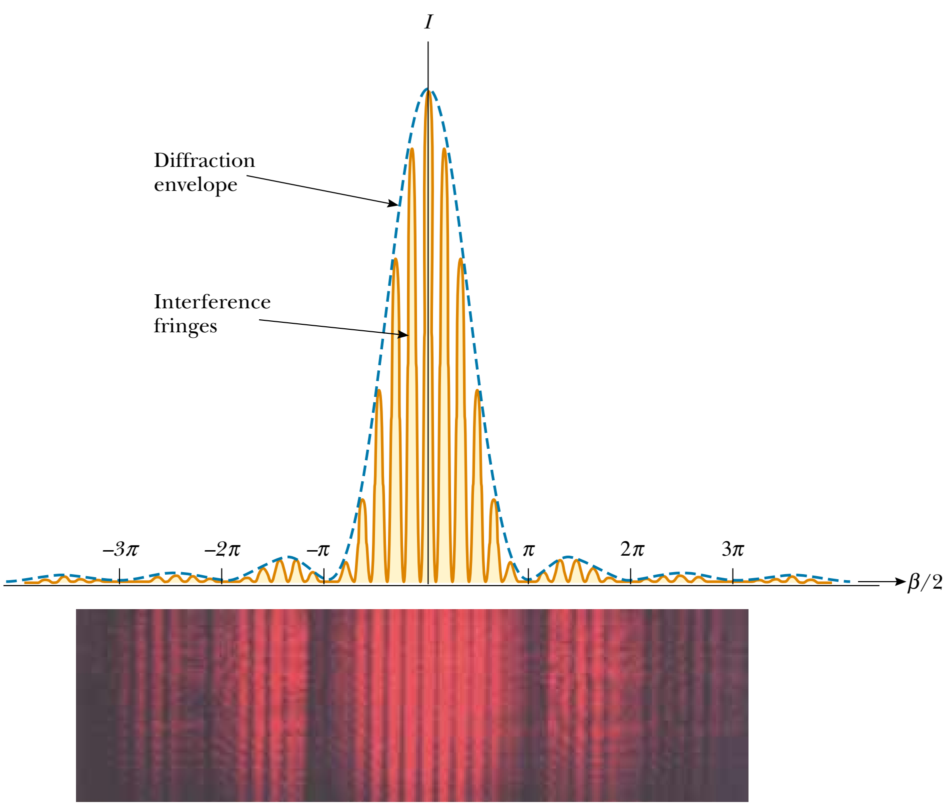


Figure 38.11 The combined effects of diffraction and interference. This is the pattern produced when 650-nm light waves pass through two 3.0-μm slits that are 18 μm apart. Notice how the diffraction pattern acts as an "envelope" and controls the intensity of the regularly spaced interference maxima.

Quick Quiz 38.2

Using Figure 38.11 as a starting point, make a sketch of the combined diffraction and interference pattern for 650-nm light waves striking two 3.0-μm slits located 9.0 μm apart.

38.3 RESOLUTION OF SINGLE-SLIT AND CIRCULAR APERTURES

The ability of optical systems to distinguish between closely spaced objects is limited because of the wave nature of light. To understand this difficulty, let us consider Figure 38.12, which shows two light sources far from a narrow slit of width a. The sources can be considered as two noncoherent point sources S_1 and S_2—for example, they could be two distant stars. If no diffraction occurred, two distinct bright spots (or images) would be observed on the viewing screen. However, because of diffraction, each source is imaged as a bright central region flanked by weaker bright and dark fringes. What is observed on the screen is the sum of two diffraction patterns: one from S_1, and the other from S_2.

If the two sources are far enough apart to keep their central maxima from overlapping, as shown in Figure 38.12a, their images can be distinguished and are said to be *resolved*. If the sources are close together, however, as shown in Figure 38.12b, the two central maxima overlap, and the images are not resolved. In determining whether two images are resolved, the following condition is often used:

When the central maximum of one image falls on the first minimum of the other image, the images are said to be just resolved. This limiting condition of resolution is known as **Rayleigh's criterion.**

Figure 38.13 shows diffraction patterns for three situations. When the objects are far apart, their images are well resolved (Fig. 38.13a). When the angular sepa-

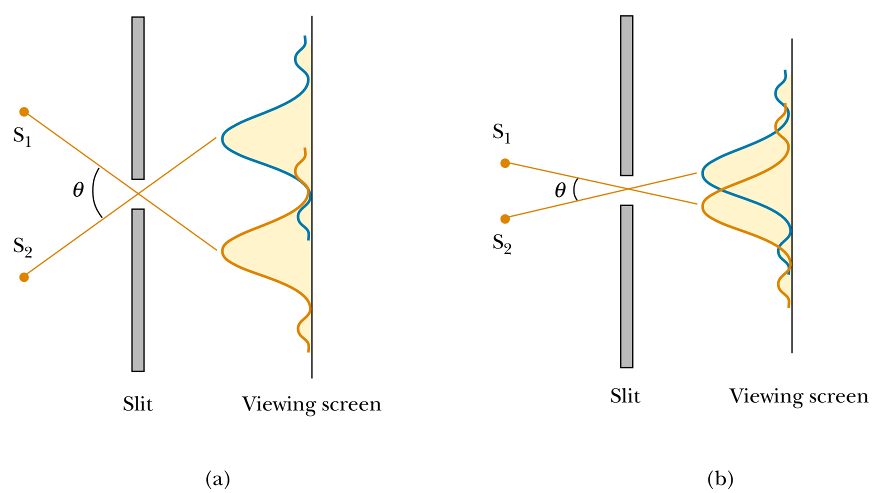


Figure 38.12 Two point sources far from a narrow slit each produce a diffraction pattern. (a) The angle subtended by the sources at the slit is large enough for the diffraction patterns to be distinguishable. (b) The angle subtended by the sources is so small that their diffraction patterns overlap, and the images are not well resolved. (Note that the angles are greatly exaggerated. The drawing is not to scale.)

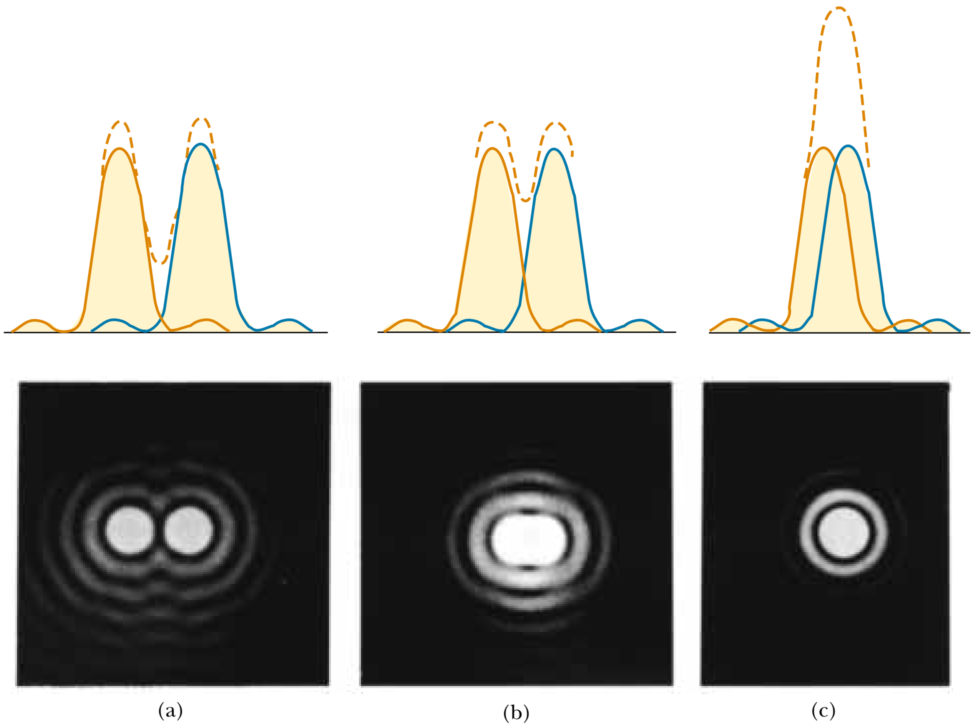


Figure 38.13 Individual diffraction patterns of two point sources (solid curves) and the resultant patterns (dashed curves) for various angular separations of the sources. In each case, the dashed curve is the sum of the two solid curves. (a) The sources are far apart, and the patterns are well resolved. (b) The sources are closer together such that the angular separation just satisfies Rayleigh's criterion, and the patterns are just resolved. (c) The sources are so close together that the patterns are not resolved.

ration of the objects satisfies Rayleigh's criterion (Fig. 38.13b), the images are just resolved. Finally, when the objects are close together, the images are not resolved (Fig. 38.13c).

From Rayleigh's criterion, we can determine the minimum angular separation θ_{min} subtended by the sources at the slit for which the images are just resolved. Equation 38.1 indicates that the first minimum in a single-slit diffraction pattern occurs at the angle for which

$$\sin\theta = \frac{\lambda}{a}$$

where a is the width of the slit. According to Rayleigh's criterion, this expression gives the smallest angular separation for which the two images are resolved. Because $\lambda \ll a$ in most situations, $\sin\theta$ is small, and we can use the approximation $\sin\theta \approx \theta$. Therefore, the limiting angle of resolution for a slit of width a is

$$\theta_{\text{min}} = \frac{\lambda}{a} \tag{38.8}$$

where θ_{min} is expressed in radians. Hence, the angle subtended by the two sources at the slit must be greater than λ/a if the images are to be resolved.

Many optical systems use circular apertures rather than slits. The diffraction pattern of a circular aperture, shown in Figure 38.14, consists of a central circular

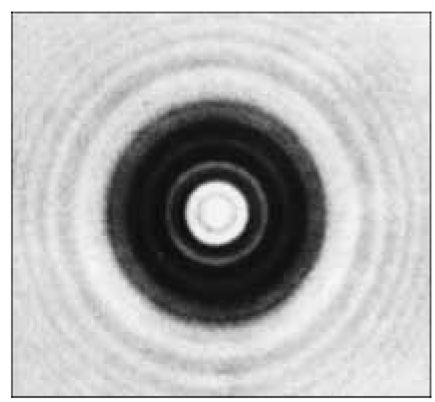


Figure 38.14 The diffraction pattern of a circular aperture consists of a central bright disk surrounded by concentric bright and dark rings.

bright disk surrounded by progressively fainter bright and dark rings. Analysis shows that the limiting angle of resolution of the circular aperture is

Limiting angle of resolution for a circular aperture

$$\theta_{\min} = 1.22 \frac{\lambda}{D} \tag{38.9}$$

where D is the diameter of the aperture. Note that this expression is similar to Equation 38.8 except for the factor 1.22, which arises from a complex mathematical analysis of diffraction from the circular aperture.

EXAMPLE 38.3 Limiting Resolution of a Microscope

Light of wavelength 589 nm is used to view an object under a microscope. If the aperture of the objective has a diameter of 0.900 cm, (a) what is the limiting angle of resolution?

Solution (a) Using Equation 38.9, we find that the limiting angle of resolution is

$$\theta_{\min} = 1.22\left(\frac{589 \times 10^{-9}\text{ m}}{0.900 \times 10^{-2}\text{ m}}\right) = \boxed{7.98 \times 10^{-5}\text{ rad}}$$

This means that any two points on the object subtending an angle smaller than this at the objective cannot be distinguished in the image.

(b) If it were possible to use visible light of any wavelength, what would be the maximum limit of resolution for this microscope?

Solution To obtain the smallest limiting angle, we have to use the shortest wavelength available in the visible spectrum. Violet light (400 nm) gives a limiting angle of resolution of

$$\theta_{\min} = 1.22\left(\frac{400 \times 10^{-9}\text{ m}}{0.900 \times 10^{-2}\text{ m}}\right) = \boxed{5.42 \times 10^{-5}\text{ rad}}$$

(c) Suppose that water ($n = 1.33$) fills the space between the object and the objective. What effect does this have on resolving power when 589-nm light is used?

Solution We find the wavelength of the 589-nm light in the water using Equation 35.7:

$$\lambda_{\text{water}} = \frac{\lambda_{\text{air}}}{n_{\text{water}}} = \frac{589\text{ nm}}{1.33} = 443\text{ nm}$$

The limiting angle of resolution at this wavelength is now smaller than that calculated in part (a):

$$\theta_{\min} = 1.22\left(\frac{443 \times 10^{-9}\text{ m}}{0.900 \times 10^{-2}\text{ m}}\right) = \boxed{6.00 \times 10^{-5}\text{ rad}}$$

EXAMPLE 38.4 Resolution of a Telescope

The Hale telescope at Mount Palomar has a diameter of 200 in. What is its limiting angle of resolution for 600-nm light?

Solution Because $D = 200$ in. $= 5.08$ m and $\lambda = 6.00 \times 10^{-7}$ m, Equation 38.9 gives

$$\theta_{\min} = 1.22 \frac{\lambda}{D} = 1.22\left(\frac{6.00 \times 10^{-7}\text{ m}}{5.08\text{ m}}\right)$$

$$= \boxed{1.44 \times 10^{-7}\text{ rad} \approx 0.03\text{ s of arc}}$$

Any two stars that subtend an angle greater than or equal to this value are resolved (if atmospheric conditions are ideal).

The Hale telescope can never reach its diffraction limit because the limiting angle of resolution is always set by atmospheric blurring. This seeing limit is usually about 1 s of arc and is never smaller than about 0.1 s of arc. (This is one of the reasons for the superiority of photographs from the Hubble Space Telescope, which views celestial objects from an orbital position above the atmosphere.)

Exercise The large radio telescope at Arecibo, Puerto Rico, has a diameter of 305 m and is designed to detect 0.75-m radio waves. Calculate the minimum angle of resolution for this telescope and compare your answer with that for the Hale telescope.

Answer 3.0×10^{-3} rad (10 min of arc), more than 10 000 times larger (that is, *worse*) than the Hale minimum.

EXAMPLE 38.5 Resolution of the Eye

Estimate the limiting angle of resolution for the human eye, assuming its resolution is limited only by diffraction.

Solution Let us choose a wavelength of 500 nm, near the center of the visible spectrum. Although pupil diameter

varies from person to person, we estimate a diameter of 2 mm. We use Equation 38.9, taking $\lambda = 500$ nm and $D = 2$ mm:

$$\theta_{\min} = 1.22\,\frac{\lambda}{D} = 1.22\left(\frac{5.00 \times 10^{-7}\text{ m}}{2 \times 10^{-3}\text{ m}}\right)$$

$$\approx \boxed{3 \times 10^{-4}\text{ rad} \approx 1\text{ min of arc}}$$

We can use this result to determine the minimum separation distance d between two point sources that the eye can distinguish if they are a distance L from the observer (Fig. 38.15). Because $\theta_{\min}$ is small, we see that

$$\sin\theta_{\min} \approx \theta_{\min} \approx \frac{d}{L}$$

$$d = L\theta_{\min}$$

For example, if the point sources are 25 cm from the eye (the near point), then

$$d = (25\text{ cm})(3 \times 10^{-4}\text{ rad}) = 8 \times 10^{-3}\text{ cm}$$

This is approximately equal to the thickness of a human hair.

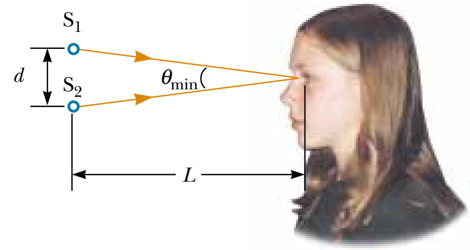


Figure 38.15 Two point sources separated by a distance d as observed by the eye.

Exercise Suppose that the pupil is dilated to a diameter of 5.0 mm and that two point sources 3.0 m away are being viewed. How far apart must the sources be if the eye is to resolve them?

Answer 0.037 cm.

APPLICATION Loudspeaker Design

The three-way speaker system shown in Figure 38.16 contains a woofer, a midrange speaker, and a tweeter. The small-diameter tweeter is for high frequencies, and the large-diameter woofer is for low frequencies. The midrange speaker, of intermediate diameter, is used for the frequency band above the high-frequency cutoff of the woofer and below the low-frequency cutoff of the tweeter. Circuits known as crossover networks include low-pass, midrange, and high-pass filters that direct the electrical signal to the appropriate speaker. The effective aperture size of a speaker is approximately its diameter. Because the wavelengths of sound waves are comparable to the typical sizes of the speakers, diffraction effects determine the angular radiation pattern. To be most useful, a speaker should radiate sound over a broad range of angles so that the listener does not have to stand at a particular spot in the room to hear maximum sound intensity. On the basis of the angular radiation pattern, let us investigate the frequency range for which a 6-in. (0.15-m) midrange speaker is most useful.

The speed of sound in air is 344 m/s, and for a circular aperture, diffraction effects become important when $\lambda = 1.22D$, where D is the speaker diameter. Therefore, we would expect this speaker to radiate non-uniformly for all frequencies above

$$\frac{344\text{ m/s}}{1.22(0.15\text{ m})} = 1\,900\text{ Hz}$$

Suppose our design specifies that the midrange speaker operates between 500 Hz (the high-frequency woofer cutoff) and 2 000 Hz. Measurements of the dispersion of radiated

Figure 38.16 An audio speaker system for high-fidelity sound reproduction. The tweeter is at the top, the midrange speaker is in the middle, and the woofer is at the bottom. *(International Stock Photography)*

sound at a suitably great distance from the speaker yield the angular profiles of sound intensity shown in Figure 38.17. In examining these plots, we see that the dispersion pattern for a 500-Hz sound is fairly uniform. This angular range is sufficiently great for us to say that this midrange speaker satisfies the design criterion. The intensity of a 2 000-Hz sound decreases to about half its maximum value about 30° from the centerline.

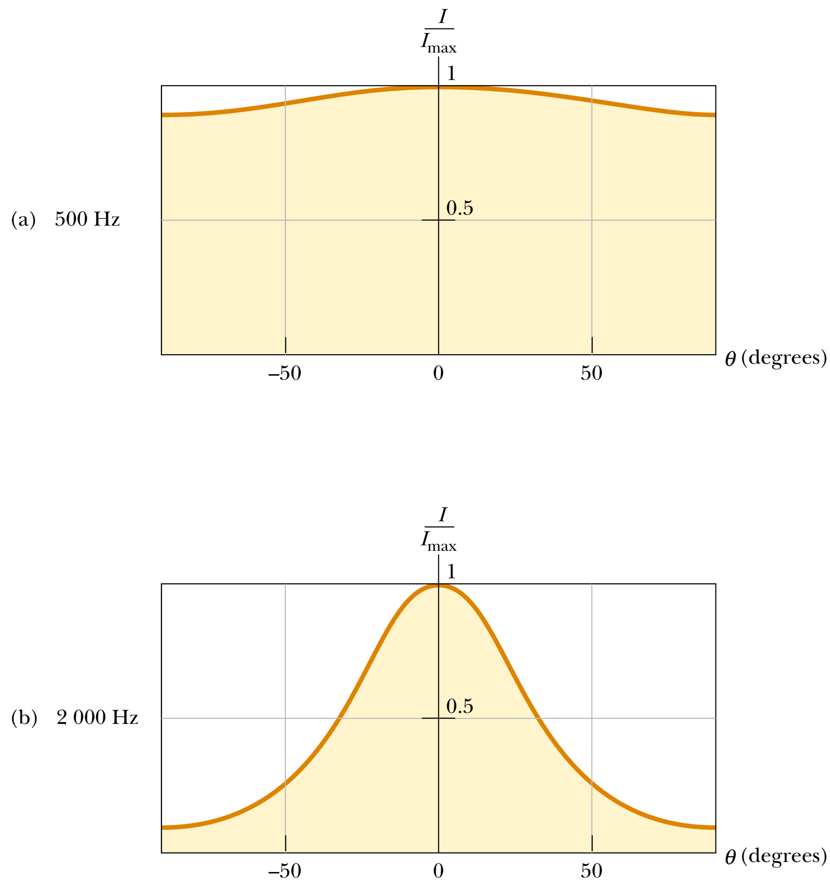


Figure 38.17 Angular dispersion of sound intensity I for a midrange speaker at (a) 500 Hz and (b) 2 000 Hz.

38.4 THE DIFFRACTION GRATING

The **diffraction grating,** a useful device for analyzing light sources, consists of a large number of equally spaced parallel slits. A *transmission grating* can be made by cutting parallel lines on a glass plate with a precision ruling machine. The spaces between the lines are transparent to the light and hence act as separate slits. A *reflection grating* can be made by cutting parallel lines on the surface of a reflective material. The reflection of light from the spaces between the lines is specular, and the reflection from the lines cut into the material is diffuse. Thus, the spaces between the lines act as parallel sources of reflected light, like the slits in a transmission grating. Gratings that have many lines very close to each other can have very small slit spacings. For example, a grating ruled with 5 000 lines/cm has a slit spacing $d = (1/5\,000)$ cm $= 2.00 \times 10^{-4}$ cm.

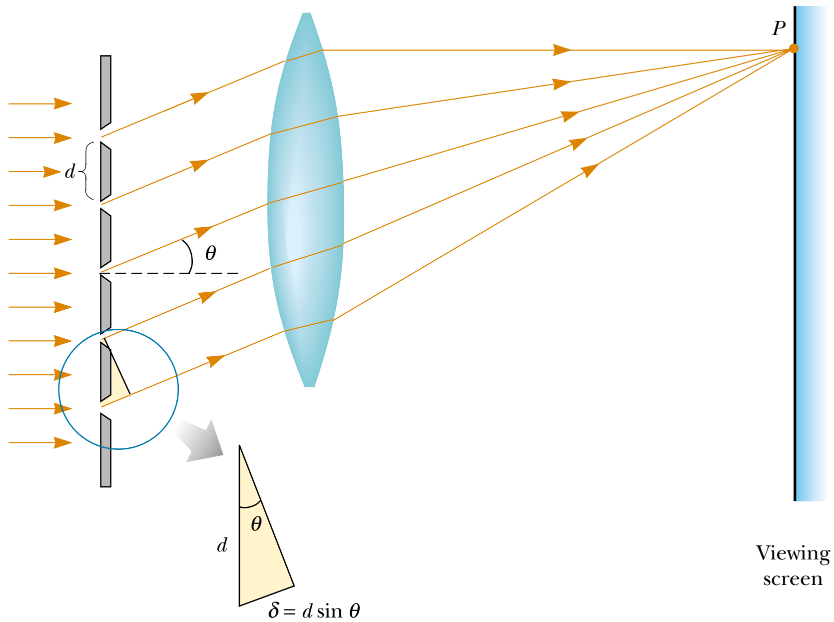


Figure 38.18 Side view of a diffraction grating. The slit separation is d, and the path difference between adjacent slits is $d \sin \theta$.

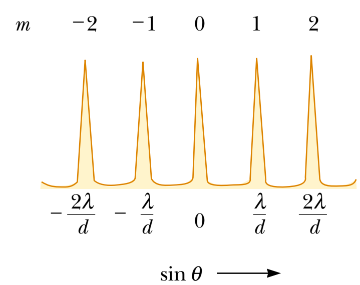


Figure 38.19 Intensity versus $\sin \theta$ for a diffraction grating. The zeroth-, first-, and second-order maxima are shown.

A section of a diffraction grating is illustrated in Figure 38.18. A plane wave is incident from the left, normal to the plane of the grating. A converging lens brings the rays together at point P. The pattern observed on the screen is the result of the combined effects of interference and diffraction. Each slit produces diffraction, and the diffracted beams interfere with one another to produce the final pattern.

The waves from all slits are in phase as they leave the slits. However, for some arbitrary direction θ measured from the horizontal, the waves must travel different path lengths before reaching point P. From Figure 38.18, note that the path difference δ between rays from any two adjacent slits is equal to $d \sin \theta$. If this path difference equals one wavelength or some integral multiple of a wavelength, then waves from all slits are in phase at point P and a bright fringe is observed. Therefore, the condition for maxima in the interference pattern at the angle θ is

Condition for interference maxima for a grating

$$d \sin \theta = m\lambda \qquad m = 0, 1, 2, 3, \ldots \qquad \textbf{(38.10)}$$

We can use this expression to calculate the wavelength if we know the grating spacing and the angle θ. If the incident radiation contains several wavelengths, the mth-order maximum for each wavelength occurs at a specific angle. All wavelengths are seen at $\theta = 0$, corresponding to $m = 0$, the zeroth-order maximum. The first-order maximum ($m = 1$) is observed at an angle that satisfies the relationship $\sin \theta = \lambda/d$; the second-order maximum ($m = 2$) is observed at a larger angle θ, and so on.

The intensity distribution for a diffraction grating obtained with the use of a monochromatic source is shown in Figure 38.19. Note the sharpness of the principal maxima and the broadness of the dark areas. This is in contrast to the broad bright fringes characteristic of the two-slit interference pattern (see Fig. 37.6). Because the principal maxima are so sharp, they are very much brighter than two-slit

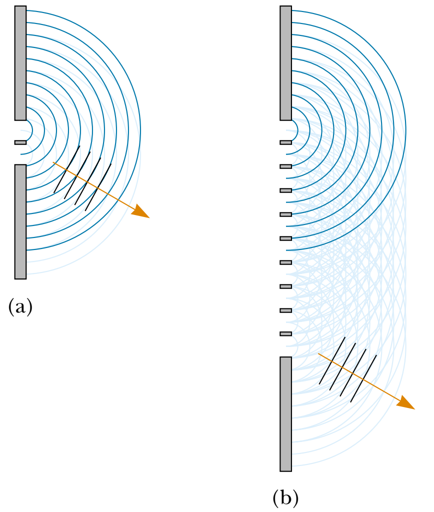


Figure 38.20 (a) Addition of two wave fronts from two slits. (b) Addition of ten wave fronts from ten slits. The resultant wave is much stronger in part (b) than in part (a).

QuickLab

Stand a couple of meters from a lightbulb. Facing away from the light, hold a compact disc about 10 cm from your eye and tilt it until the reflection of the bulb is located in the hole at the disc's center. You should see spectra radiating out from the center, with violet on the inside and red on the outside. Now move the disc away from your eye until the violet band is at the outer edge. Carefully measure the distance from your eye to the center of the disc and also determine the radius of the disc. Use this information to find the angle θ to the first-order maximum for violet light. Now use Equation 38.10 to determine the spacing between the grooves on the disc. The industry standard is 1.6 μm. How close did you come?

interference maxima. The reason for this is illustrated in Figure 38.20, in which the combination of multiple wave fronts for a ten-slit grating is compared with the wave fronts for a two-slit system. Actual gratings have thousands of times more slits, and therefore the maxima are even stronger.

A schematic drawing of a simple apparatus used to measure angles in a diffraction pattern is shown in Figure 38.21. This apparatus is a diffraction grating spectrometer. The light to be analyzed passes through a slit, and a collimated beam of light is incident on the grating. The diffracted light leaves the grating at angles that satisfy Equation 38.10, and a telescope is used to view the image of the slit. The wavelength can be determined by measuring the precise angles at which the images of the slit appear for the various orders.

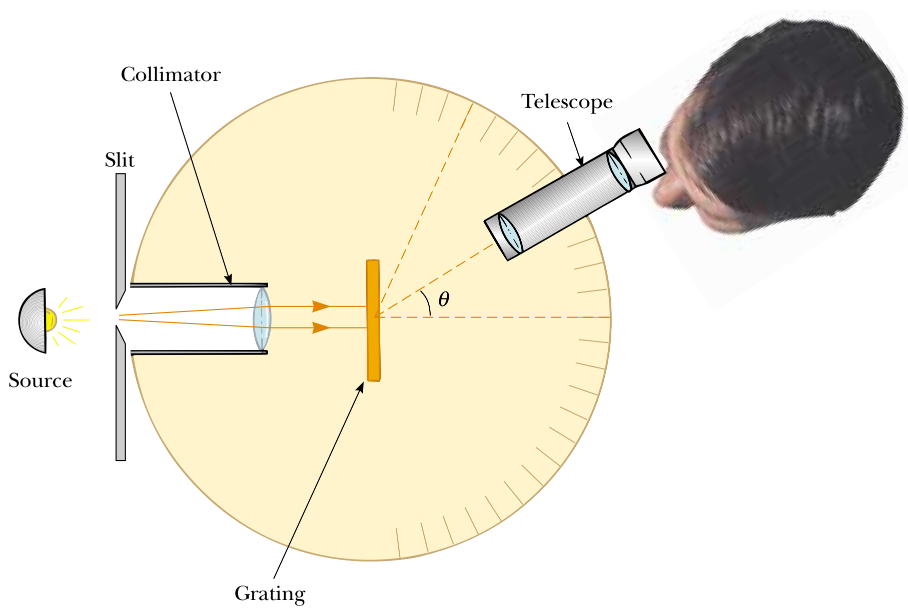


Figure 38.21 Diagram of a diffraction grating spectrometer. The collimated beam incident on the grating is diffracted into the various orders at the angles θ that satisfy the equation $d \sin \theta = m\lambda$, where $m = 0, 1, 2, \ldots$.

CONCEPTUAL EXAMPLE 38.6 A Compact Disc Is a Diffraction Grating

Light reflected from the surface of a compact disc is multicolored, as shown in Figure 38.22. The colors and their intensities depend on the orientation of the disc relative to the eye and relative to the light source. Explain how this works.

Solution The surface of a compact disc has a spiral grooved track (with adjacent grooves having a separation on the order of 1 μm). Thus, the surface acts as a reflection grating. The light reflecting from the regions between these closely spaced grooves interferes constructively only in certain directions that depend on the wavelength and on the direction of the incident light. Any one section of the disc serves as a diffraction grating for white light, sending different colors in different directions. The different colors you see when viewing one section change as the light source, the disc, or you move to change the angles of incidence or diffraction.

Figure 38.22 A compact disc observed under white light. The colors observed in the reflected light and their intensities depend on the orientation of the disc relative to the eye and relative to the light source.

EXAMPLE 38.7 The Orders of a Diffraction Grating

Monochromatic light from a helium-neon laser ($\lambda = 632.8$ nm) is incident normally on a diffraction grating containing 6 000 lines per centimeter. Find the angles at which the first-order, second-order, and third-order maxima are observed.

Solution First, we must calculate the slit separation, which is equal to the inverse of the number of lines per centimeter:

$$d = \frac{1}{6\,000}\text{ cm} = 1.667 \times 10^{-4}\text{ cm} = 1\,667\text{ nm}$$

For the first-order maximum ($m = 1$), we obtain

$$\sin\theta_1 = \frac{\lambda}{d} = \frac{632.8\text{ nm}}{1\,667\text{ nm}} = 0.379\,6$$

$$\theta_1 = \boxed{22.31°}$$

For the second-order maximum ($m = 2$), we find

$$\sin\theta_2 = \frac{2\lambda}{d} = \frac{2(632.8\text{ nm})}{1\,667\text{ nm}} = 0.759\,2$$

$$\theta_2 = \boxed{49.39°}$$

For $m = 3$, we find that $\sin\theta_3 = 1.139$. Because $\sin\theta$ cannot exceed unity, this does not represent a realistic solution. Hence, only zeroth-, first-, and second-order maxima are observed for this situation.

Resolving Power of the Diffraction Grating

The diffraction grating is most useful for measuring wavelengths accurately. Like the prism, the diffraction grating can be used to disperse a spectrum into its wavelength components. Of the two devices, the grating is the more precise if one wants to distinguish two closely spaced wavelengths.

For two nearly equal wavelengths λ_1 and λ_2 between which a diffraction grating can just barely distinguish, the **resolving power** R of the grating is defined as

$$R = \frac{\lambda}{\lambda_2 - \lambda_1} = \frac{\lambda}{\Delta\lambda} \qquad \textbf{(38.11)}$$

Resolving power

where $\lambda = (\lambda_1 + \lambda_2)/2$ and $\Delta\lambda = \lambda_2 - \lambda_1$. Thus, a grating that has a high resolving power can distinguish small differences in wavelength. If N lines of the grating

are illuminated, it can be shown that the resolving power in the mth-order diffraction is

Resolving power of a grating

$$R = Nm \quad \textbf{(38.12)}$$

Thus, resolving power increases with increasing order number and with increasing number of illuminated slits.

Note that $R = 0$ for $m = 0$; this signifies that all wavelengths are indistinguishable for the zeroth-order maximum. However, consider the second-order diffraction pattern ($m = 2$) of a grating that has 5 000 rulings illuminated by the light source. The resolving power of such a grating in second order is $R = 5\,000 \times 2 = 10\,000$. Therefore, for a mean wavelength of, for example, 600 nm, the minimum wavelength separation between two spectral lines that can be just resolved is $\Delta\lambda = \lambda/R = 6.00 \times 10^{-2}$ nm. For the third-order principal maximum, $R = 15\,000$ and $\Delta\lambda = 4.00 \times 10^{-2}$ nm, and so on.

One of the most interesting applications of diffraction is holography, which is used to create three-dimensional images found practically everywhere, from credit cards to postage stamps. The production of these special diffracting films is discussed in Chapter 42 of the extended version of this text.

EXAMPLE 38.8 Resolving Sodium Spectral Lines

When an element is raised to a very high temperature, the atoms emit radiation having discrete wavelengths. The set of wavelengths for a given element is called its *atomic spectrum.* Two strong components in the atomic spectrum of sodium have wavelengths of 589.00 nm and 589.59 nm. (a) What must be the resolving power of a grating if these wavelengths are to be distinguished?

Solution

$$R = \frac{\lambda}{\Delta\lambda} = \frac{589.30 \text{ nm}}{589.59 \text{ nm} - 589.00 \text{ nm}} = \frac{589.30}{0.59} = \boxed{999}$$

(b) To resolve these lines in the second-order spectrum, how many lines of the grating must be illuminated?

Solution From Equation 38.12 and the results to part (a), we find that

$$N = \frac{R}{m} = \frac{999}{2} = \boxed{500 \text{ lines}}$$

Optional Section

38.5 DIFFRACTION OF X-RAYS BY CRYSTALS

In principle, the wavelength of any electromagnetic wave can be determined if a grating of the proper spacing (of the order of λ) is available. X-rays, discovered by Wilhelm Roentgen (1845–1923) in 1895, are electromagnetic waves of very short wavelength (of the order of 0.1 nm). It would be impossible to construct a grating having such a small spacing by the cutting process described at the beginning of Section 38.4. However, the atomic spacing in a solid is known to be about 0.1 nm. In 1913, Max von Laue (1879–1960) suggested that the regular array of atoms in a crystal could act as a three-dimensional diffraction grating for x-rays. Subsequent experiments confirmed this prediction. The diffraction patterns are complex because of the three-dimensional nature of the crystal. Nevertheless, x-ray diffraction

has proved to be an invaluable technique for elucidating crystalline structures and for understanding the structure of matter.[1]

Figure 38.23 is one experimental arrangement for observing x-ray diffraction from a crystal. A collimated beam of x-rays is incident on a crystal. The diffracted beams are very intense in certain directions, corresponding to constructive interference from waves reflected from layers of atoms in the crystal. The diffracted beams can be detected by a photographic film, and they form an array of spots known as a *Laue pattern*. One can deduce the crystalline structure by analyzing the positions and intensities of the various spots in the pattern.

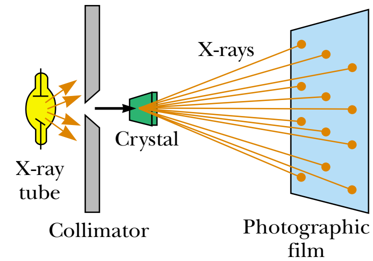


Figure 38.23 Schematic diagram of the technique used to observe the diffraction of x-rays by a crystal. The array of spots formed on the film is called a Laue pattern.

The arrangement of atoms in a crystal of sodium chloride (NaCl) is shown in Figure 38.24. Each unit cell (the geometric solid that repeats throughout the crystal) is a cube having an edge length a. A careful examination of the NaCl structure shows that the ions lie in discrete planes (the shaded areas in Fig. 38.24). Now suppose that an incident x-ray beam makes an angle θ with one of the planes, as shown in Figure 38.25. The beam can be reflected from both the upper plane and the lower one. However, the beam reflected from the lower plane travels farther than the beam reflected from the upper plane. The effective path difference is $2d \sin \theta$. The two beams reinforce each other (constructive interference) when this path difference equals some integer multiple of λ. The same is true for reflection from the entire family of parallel planes. Hence, the condition for constructive interference (maxima in the reflected beam) is

Bragg's law

$$2d \sin \theta = m\lambda \qquad m = 1, 2, 3, \ldots \tag{38.13}$$

This condition is known as **Bragg's law,** after W. L. Bragg (1890–1971), who first derived the relationship. If the wavelength and diffraction angle are measured, Equation 38.13 can be used to calculate the spacing between atomic planes.

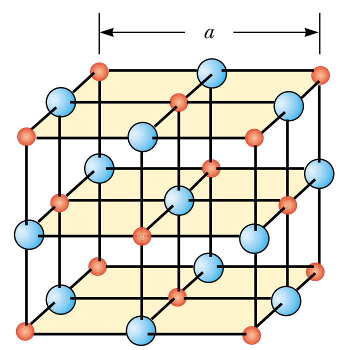


Figure 38.24 Crystalline structure of sodium chloride (NaCl). The blue spheres represent Cl^- ions, and the red spheres represent Na^+ ions. The length of the cube edge is $a = 0.562\ 737$ nm.

Quick Quiz 38.3

When you receive a chest x-ray at a hospital, the rays pass through a series of parallel ribs in your chest. Do the ribs act as a diffraction grating for x-rays?

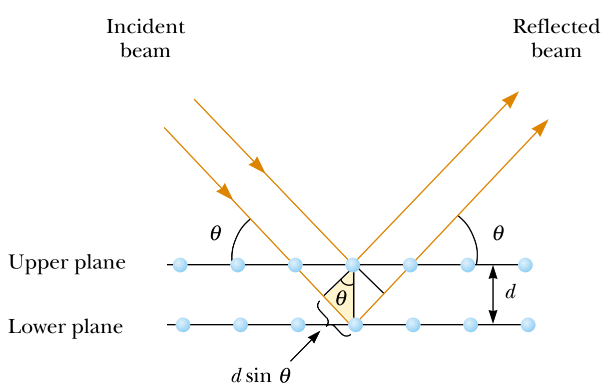


Figure 38.25 A two-dimensional description of the reflection of an x-ray beam from two parallel crystalline planes separated by a distance d. The beam reflected from the lower plane travels farther than the one reflected from the upper plane by a distance $2d \sin \theta$.

[1] For more details on this subject, see Sir Lawrence Bragg, "X-Ray Crystallography," *Sci. Am.* 219:58–70, 1968.

38.6 POLARIZATION OF LIGHT WAVES

In Chapter 34 we described the transverse nature of light and all other electromagnetic waves. Polarization is firm evidence of this transverse nature.

An ordinary beam of light consists of a large number of waves emitted by the atoms of the light source. Each atom produces a wave having some particular orientation of the electric field vector **E**, corresponding to the direction of atomic vibration. The *direction of polarization* of each individual wave is defined to be the direction in which the electric field is vibrating. In Figure 38.26, this direction happens to lie along the y axis. However, an individual electromagnetic wave could have its **E** vector in the yz plane, making any possible angle with the y axis. Because all directions of vibration from a wave source are possible, the resultant electromagnetic wave is a superposition of waves vibrating in many different directions. The result is an **unpolarized** light beam, represented in Figure 38.27a. The direction of wave propagation in this figure is perpendicular to the page. The arrows show a few possible directions of the electric field vectors for the individual waves making up the resultant beam. At any given point and at some instant of time, all these individual electric field vectors add to give one resultant electric field vector.

As noted in Section 34.2, a wave is said to be **linearly polarized** if the resultant electric field **E** vibrates in the same direction *at all times* at a particular point, as shown in Figure 38.27b. (Sometimes, such a wave is described as *plane-polarized,* or simply *polarized.*) The plane formed by **E** and the direction of propagation is called the *plane of polarization* of the wave. If the wave in Figure 38.26 represented the resultant of all individual waves, the plane of polarization is the xy plane.

It is possible to obtain a linearly polarized beam from an unpolarized beam by removing all waves from the beam except those whose electric field vectors oscillate in a single plane. We now discuss four processes for producing polarized light from unpolarized light.

Polarization by Selective Absorption

The most common technique for producing polarized light is to use a material that transmits waves whose electric fields vibrate in a plane parallel to a certain direction and that absorbs waves whose electric fields vibrate in all other directions.

In 1938, E. H. Land (1909–1991) discovered a material, which he called *polaroid,* that polarizes light through selective absorption by oriented molecules. This material is fabricated in thin sheets of long-chain hydrocarbons. The sheets are stretched during manufacture so that the long-chain molecules align. After a sheet is dipped into a solution containing iodine, the molecules become good electrical conductors. However, conduction takes place primarily along the hydrocarbon chains because electrons can move easily only along the chains. As a result, the

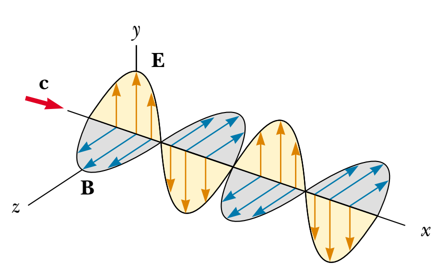


Figure 38.26 Schematic diagram of an electromagnetic wave propagating at velocity **c** in the x direction. The electric field vibrates in the xy plane, and the magnetic field vibrates in the xz plane.

molecules readily absorb light whose electric field vector is parallel to their length and allow light through whose electric field vector is perpendicular to their length.

It is common to refer to the direction perpendicular to the molecular chains as the *transmission axis.* In an ideal polarizer, all light with **E** parallel to the transmission axis is transmitted, and all light with **E** perpendicular to the transmission axis is absorbed.

Figure 38.28 represents an unpolarized light beam incident on a first polarizing sheet, called the *polarizer.* Because the transmission axis is oriented vertically in the figure, the light transmitted through this sheet is polarized vertically. A second polarizing sheet, called the *analyzer,* intercepts the beam. In Figure 38.28, the analyzer transmission axis is set at an angle θ to the polarizer axis. We call the electric field vector of the transmitted beam $\mathbf{E}_0$. The component of $\mathbf{E}_0$ perpendicular to the analyzer axis is completely absorbed. The component of $\mathbf{E}_0$ parallel to the analyzer axis, which is allowed through by the analyzer, is $E_0 \cos \theta$. Because the intensity of the transmitted beam varies as the square of its magnitude, we conclude that the intensity of the (polarized) beam transmitted through the analyzer varies as

$$I = I_{\text{max}} \cos^2 \theta \qquad \textbf{(38.14)}$$

where I_{max} is the intensity of the polarized beam incident on the analyzer. This expression, known as **Malus's law,**[2] applies to any two polarizing materials whose transmission axes are at an angle θ to each other. From this expression, note that the intensity of the transmitted beam is maximum when the transmission axes are parallel ($\theta = 0$ or $180°$) and that it is zero (complete absorption by the analyzer) when the transmission axes are perpendicular to each other. This variation in transmitted intensity through a pair of polarizing sheets is illustrated in Figure 38.29. Because the average value of $\cos^2 \theta$ is $\frac{1}{2}$, the intensity of the light passed through an ideal polarizer is one-half the intensity of unpolarized light.

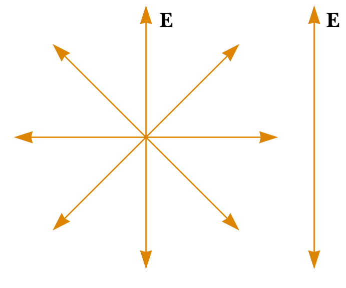


Figure 38.27 (a) An unpolarized light beam viewed along the direction of propagation (perpendicular to the page). The transverse electric field can vibrate in any direction in the plane of the page with equal probability. (b) A linearly polarized light beam with the electric field vibrating in the vertical direction.

Polarization by Reflection

When an unpolarized light beam is reflected from a surface, the reflected light may be completely polarized, partially polarized, or unpolarized, depending on the angle of incidence. If the angle of incidence is 0°, the reflected beam is unpolarized. For other angles of incidence, the reflected light is polarized to some ex-

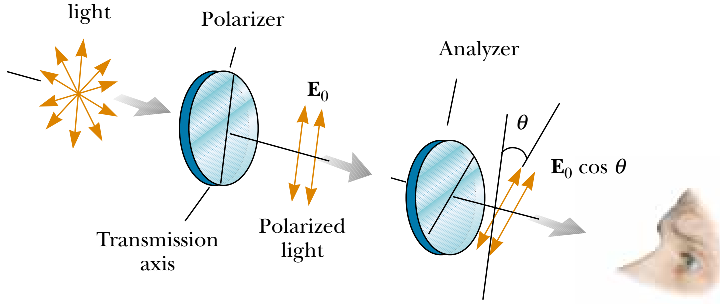


Figure 38.28 Two polarizing sheets whose transmission axes make an angle θ with each other. Only a fraction of the polarized light incident on the analyzer is transmitted through it.

[2] Named after its discoverer, E. L. Malus (1775–1812). Malus discovered that reflected light was polarized by viewing it through a calcite ($CaCO_3$) crystal.

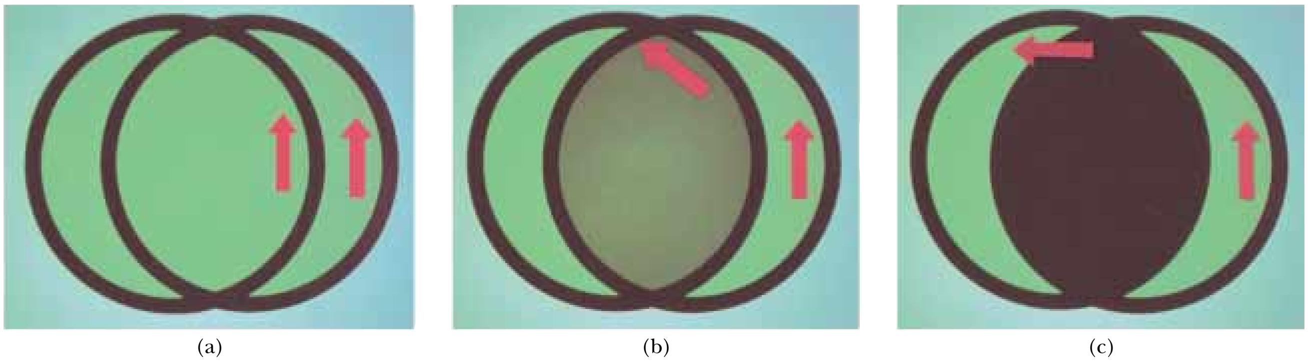


Figure 38.29 The intensity of light transmitted through two polarizers depends on the relative orientation of their transmission axes. (a) The transmitted light has maximum intensity when the transmission axes are aligned with each other. (b) The transmitted light has lesser intensity when the transmission axes are at an angle of 45° with each other. (c) The transmitted light intensity is a minimum when the transmission axes are at right angles to each other.

tent, and for one particular angle of incidence, the reflected light is completely polarized. Let us now investigate reflection at that special angle.

Suppose that an unpolarized light beam is incident on a surface, as shown in Figure 38.30a. Each individual electric field vector can be resolved into two components: one parallel to the surface (and perpendicular to the page in Fig. 38.30, represented by the dots), and the other (represented by the red arrows) perpendicular both to the first component and to the direction of propagation. Thus, the polarization of the entire beam can be described by two electric field components in these directions. It is found that the parallel component reflects more strongly than the perpendicular component, and this results in a partially polarized reflected beam. Furthermore, the refracted beam is also partially polarized.

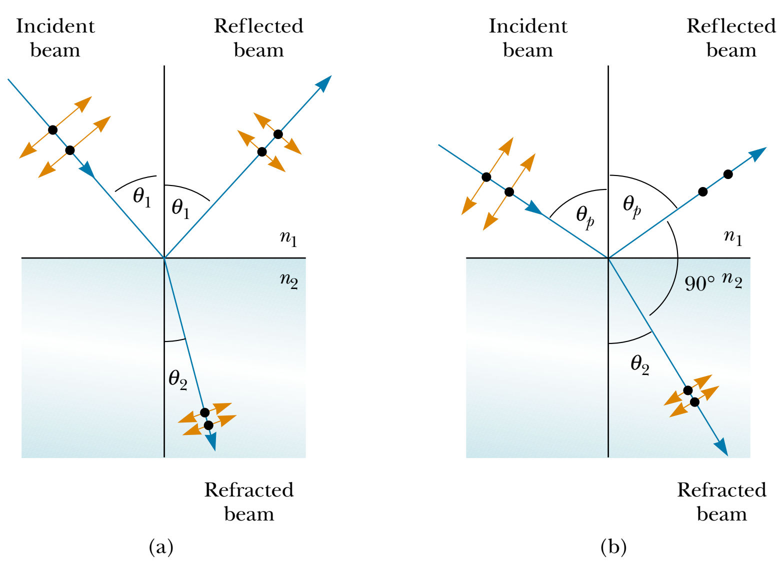


Figure 38.30 (a) When unpolarized light is incident on a reflecting surface, the reflected and refracted beams are partially polarized. (b) The reflected beam is completely polarized when the angle of incidence equals the polarizing angle θ_p, which satisfies the equation $n = \tan\theta_p$.

Now suppose that the angle of incidence θ_1 is varied until the angle between the reflected and refracted beams is 90°, as shown in Figure 38.30b. At this particular angle of incidence, the reflected beam is completely polarized (with its electric field vector parallel to the surface), and the refracted beam is still only partially polarized. The angle of incidence at which this polarization occurs is called the **polarizing angle** θ_p.

Polarizing angle

We can obtain an expression relating the polarizing angle to the index of refraction of the reflecting substance by using Figure 38.30b. From this figure, we see that $\theta_p + 90° + \theta_2 = 180°$; thus, $\theta_2 = 90° - \theta_p$. Using Snell's law of refraction (Eq. 35.8) and taking $n_1 = 1.00$ for air and $n_2 = n$, we have

$$n = \frac{\sin \theta_1}{\sin \theta_2} = \frac{\sin \theta_p}{\sin \theta_2}$$

Because $\sin \theta_2 = \sin(90° - \theta_p) = \cos \theta_p$, we can write this expression for n as $n = \sin \theta_p / \cos \theta_p$, which means that

$$n = \tan \theta_p \qquad \textbf{(38.15)}$$

Brewster's law

This expression is called **Brewster's law,** and the polarizing angle θ_p is sometimes called **Brewster's angle,** after its discoverer, David Brewster (1781–1868). Because n varies with wavelength for a given substance, Brewster's angle is also a function of wavelength.

Polarization by reflection is a common phenomenon. Sunlight reflected from water, glass, and snow is partially polarized. If the surface is horizontal, the electric field vector of the reflected light has a strong horizontal component. Sunglasses made of polarizing material reduce the glare of reflected light. The transmission axes of the lenses are oriented vertically so that they absorb the strong horizontal component of the reflected light. If you rotate sunglasses 90°, they will not be as effective at blocking the glare from shiny horizontal surfaces.

QuickLab

Devise a way to use a protractor, desklamp, and polarizing sunglasses to measure Brewster's angle for the glass in a window. From this, determine the index of refraction of the glass. Compare your results with the values given in Table 35.1.

Polarization by Double Refraction

Solids can be classified on the basis of internal structure. Those in which the atoms are arranged in a specific order are called *crystalline;* the NaCl structure of Figure 38.24 is just one example of a crystalline solid. Those solids in which the atoms are distributed randomly are called *amorphous.* When light travels through an amorphous material, such as glass, it travels with a speed that is the same in all directions. That is, glass has a single index of refraction. In certain crystalline materials, however, such as calcite and quartz, the speed of light is not the same in all directions. Such materials are characterized by two indices of refraction. Hence, they are often referred to as **double-refracting** or **birefringent** materials.

Upon entering a calcite crystal, unpolarized light splits into two plane-polarized rays that travel with different velocities, corresponding to two angles of refraction, as shown in Figure 38.31. The two rays are polarized in two mutually perpendicular directions, as indicated by the dots and arrows. One ray, called the **ordinary (O) ray,** is characterized by an index of refraction n_O that is the same in all directions. This means that if one could place a point source of light inside the crystal, as shown in Figure 38.32, the ordinary waves would spread out from the source as spheres.

The second plane-polarized ray, called the **extraordinary (E) ray,** travels with different speeds in different directions and hence is characterized by an index of refraction n_E that varies with the direction of propagation. The point source in Fig-

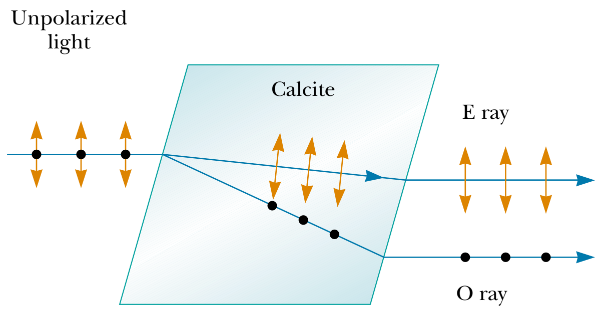


Figure 38.31 Unpolarized light incident on a calcite crystal splits into an ordinary (O) ray and an extraordinary (E) ray. These two rays are polarized in mutually perpendicular directions (drawing not to scale).

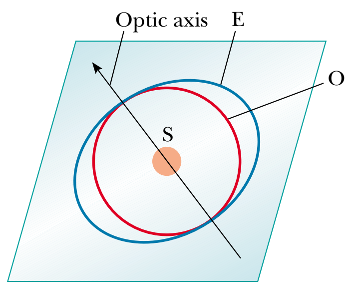


Figure 38.32 A point source S inside a double-refracting crystal produces a spherical wave front corresponding to the ordinary ray and an elliptical wave front corresponding to the extraordinary ray. The two waves propagate with the same velocity along the optic axis.

ure 38.32 sends out an extraordinary wave having wave fronts that are elliptical in cross-section. Note from Figure 38.32 that there is one direction, called the **optic axis,** along which the ordinary and extraordinary rays have the same speed, corresponding to the direction for which $n_O = n_E$. The difference in speed for the two rays is a maximum in the direction perpendicular to the optic axis. For example, in calcite, $n_O = 1.658$ at a wavelength of 589.3 nm, and n_E varies from 1.658 along the optic axis to 1.486 perpendicular to the optic axis. Values for n_O and n_E for various double-refracting crystals are given in Table 38.1.

If we place a piece of calcite on a sheet of paper and then look through the crystal at any writing on the paper, we see two images, as shown in Figure 38.33. As can be seen from Figure 38.31, these two images correspond to one formed by the ordinary ray and one formed by the extraordinary ray. If the two images are viewed through a sheet of rotating polarizing glass, they alternately appear and disappear because the ordinary and extraordinary rays are plane-polarized along mutually perpendicular directions.

Polarization by Scattering

When light is incident on any material, the electrons in the material can absorb and reradiate part of the light. Such absorption and reradiation of light by electrons in the gas molecules that make up air is what causes sunlight reaching an observer on the Earth to be partially polarized. You can observe this effect—called **scattering**—by looking directly up at the sky through a pair of sunglasses whose lenses are made of polarizing material. Less light passes through at certain orientations of the lenses than at others.

Figure 38.34 illustrates how sunlight becomes polarized when it is scattered. An unpolarized beam of sunlight traveling in the horizontal direction (parallel to

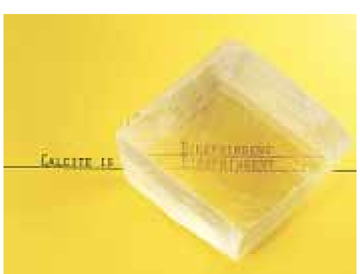


Figure 38.33 A calcite crystal produces a double image because it is a birefringent (double-refracting) material.

TABLE 38.1 Indices of Refraction for Some Double-Refracting Crystals at a Wavelength of 589.3 nm

Crystal	n_O	n_E	n_O/n_E
Calcite ($CaCO_3$)	1.658	1.486	1.116
Quartz (SiO_2)	1.544	1.553	0.994
Sodium nitrate ($NaNO_3$)	1.587	1.336	1.188
Sodium sulfite ($NaSO_3$)	1.565	1.515	1.033
Zinc chloride ($ZnCl_2$)	1.687	1.713	0.985
Zinc sulfide (ZnS)	2.356	2.378	0.991

the ground) strikes a molecule of one of the gases that make up air, setting the electrons of the molecule into vibration. These vibrating charges act like the vibrating charges in an antenna. The horizontal component of the electric field vector in the incident wave results in a horizontal component of the vibration of the charges, and the vertical component of the vector results in a vertical component of vibration. If the observer in Figure 38.34 is looking straight up (perpendicular to the original direction of propagation of the light), the vertical oscillations of the charges send no radiation toward the observer. Thus, the observer sees light that is completely polarized in the horizontal direction, as indicated by the red arrows. If the observer looks in other directions, the light is partially polarized in the horizontal direction.

Some phenomena involving the scattering of light in the atmosphere can be understood as follows. When light of various wavelengths λ is incident on gas molecules of diameter d, where $d \ll \lambda$, the relative intensity of the scattered light varies as $1/\lambda^4$. The condition $d \ll \lambda$ is satisfied for scattering from oxygen (O_2) and nitrogen (N_2) molecules in the atmosphere, whose diameters are about 0.2 nm. Hence, short wavelengths (blue light) are scattered more efficiently than long wavelengths (red light). Therefore, when sunlight is scattered by gas molecules in the air, the short-wavelength radiation (blue) is scattered more intensely than the long-wavelength radiation (red).

When you look up into the sky in a direction that is not toward the Sun, you see the scattered light, which is predominantly blue; hence, you see a blue sky. If you look toward the west at sunset (or toward the east at sunrise), you are looking in a direction toward the Sun and are seeing light that has passed through a large distance of air. Most of the blue light has been scattered by the air between you and the Sun. The light that survives this trip through the air to you has had much of its blue component scattered and is thus heavily weighted toward the red end of the spectrum; as a result, you see the red and orange colors of sunset. However, a blue sky is seen by someone to your west for whom it is still a quarter hour before sunset.

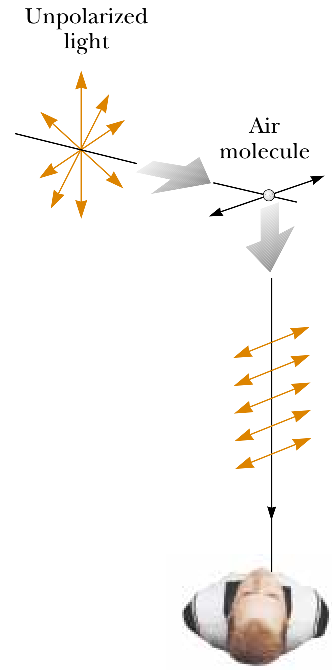


Figure 38.34 The scattering of unpolarized sunlight by air molecules. The scattered light traveling perpendicular to the incident light is plane-polarized because the vertical vibrations of the charges in the air molecule send no light in this direction.

Optical Activity

Many important applications of polarized light involve materials that display **optical activity.** A material is said to be optically active if it rotates the plane of polarization of any light transmitted through the material. The angle through which the light is rotated by a specific material depends on the length of the path through the material and on concentration if the material is in solution. One optically active material is a solution of the common sugar dextrose. A standard method for determining the concentration of sugar solutions is to measure the rotation produced by a fixed length of the solution.

Molecular asymmetry determines whether a material is optically active. For example, some proteins are optically active because of their spiral shape. Other materials, such as glass and plastic, become optically active when stressed. Suppose that an unstressed piece of plastic is placed between a polarizer and an analyzer so that light passes from polarizer to plastic to analyzer. When the plastic is unstressed and the analyzer axis is perpendicular to the polarizer axis, none of the polarized light passes through the analyzer. In other words, the unstressed plastic has no effect on the light passing through it. If the plastic is stressed, however, the regions of greatest stress rotate the polarized light through the largest angles. Hence, a series of bright and dark bands is observed in the transmitted light, with the bright bands corresponding to regions of greatest stress.

(a) (b)

Figure 38.35 (a) Strain distribution in a plastic model of a hip replacement used in a medical research laboratory. The pattern is produced when the plastic model is viewed between a polarizer and analyzer oriented perpendicular to each other. (b) A plastic model of an arch structure under load conditions observed between perpendicular polarizers. Such patterns are useful in the optimum design of architectural components.

Engineers often use this technique, called *optical stress analysis,* in designing structures ranging from bridges to small tools. They build a plastic model and analyze it under different load conditions to determine regions of potential weakness and failure under stress. Some examples of a plastic model under stress are shown in Figure 38.35.

The liquid crystal displays found in most calculators have their optical activity changed by the application of electric potential across different parts of the display. Try using a pair of polarizing sunglasses to investigate the polarization used in the display of your calculator.

SUMMARY

Diffraction is the deviation of light from a straight-line path when the light passes through an aperture or around an obstacle.

The **Fraunhofer diffraction pattern** produced by a single slit of width a on a distant screen consists of a central bright fringe and alternating bright and dark fringes of much lower intensities. The angles θ at which the diffraction pattern has zero intensity, corresponding to destructive interference, are given by

$$\sin\theta = m\frac{\lambda}{a} \qquad m = \pm 1, \pm 2, \pm 3, \ldots \tag{38.1}$$

How the intensity I of a single-slit diffraction pattern varies with angle θ is given by the expression

$$I = I_{\text{max}}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2 \tag{38.4}$$

where $\beta = (2\pi a \sin\theta)/\lambda$ and I_{max} is the intensity at $\theta = 0$.

Rayleigh's criterion, which is a limiting condition of resolution, states that two images formed by an aperture are just distinguishable if the central maximum of the diffraction pattern for one image falls on the first minimum of the diffrac-

tion pattern for the other image. The limiting angle of resolution for a slit of width a is $\theta_{\min} = \lambda/a$, and the limiting angle of resolution for a circular aperture of diameter D is $\theta_{\min} = 1.22\lambda/D$.

A **diffraction grating** consists of a large number of equally spaced, identical slits. The condition for intensity maxima in the interference pattern of a diffraction grating for normal incidence is

$$d \sin \theta = m\lambda \qquad m = 0, 1, 2, 3, \ldots \tag{38.10}$$

where d is the spacing between adjacent slits and m is the order number of the diffraction pattern. The resolving power of a diffraction grating in the mth order of the diffraction pattern is

$$R = Nm \tag{38.12}$$

where N is the number of lines in the grating that are illuminated.

When polarized light of intensity I_0 is emitted by a polarizer and then incident on an analyzer, the light transmitted through the analyzer has an intensity equal to $I_{\max} \cos^2 \theta$, where θ is the angle between the polarizer and analyzer transmission axes.

In general, reflected light is partially polarized. However, reflected light is completely polarized when the angle of incidence is such that the angle between the reflected and refracted beams is 90°. This angle of incidence, called the **polarizing angle** θ_p, satisfies **Brewster's law:**

$$n = \tan \theta_p \tag{38.15}$$

where n is the index of refraction of the reflecting medium.

Questions

1. Why can you hear around corners but not see around them?
2. Observe the shadow of your book when it is held a few inches above a table while illuminated by a lamp several feet above it. Why is the shadow somewhat fuzzy at the edges?
3. Knowing that radio waves travel at the speed of light and that a typical AM radio frequency is 1 000 kHz while an FM radio frequency might be 100 MHz, estimate the wavelengths of typical AM and FM radio signals. Use this information to explain why FM radio stations often fade out when you drive through a short tunnel or underpass but AM radio stations do not.
4. Describe the change in width of the central maximum of the single-slit diffraction pattern as the width of the slit is made narrower.
5. Assuming that the headlights of a car are point sources, estimate the maximum observer-to-car distance at which the headlights are distinguishable from each other.
6. A laser beam is incident at a shallow angle on a machinist's ruler that has a finely calibrated scale. The engraved rulings on the scale give rise to a diffraction pattern on a screen. Discuss how you can use this arrangement to obtain a measure of the wavelength of the laser light.
7. Certain sunglasses use a polarizing material to reduce the intensity of light reflected from shiny surfaces. What orientation of polarization should the material have to be most effective?
8. During the "day" on the Moon (that is, when the Sun is visible), you see a black sky and the stars are clearly visible. During the day on the Earth, you see a blue sky and no stars. Account for this difference.
9. You can make the path of a light beam visible by placing dust in the air (perhaps by shaking a blackboard eraser in the path of the light beam). Explain why you can see the beam under these circumstances.
10. Is light from the sky polarized? Why is it that clouds seen through Polaroid glasses stand out in bold contrast to the sky?
11. If a coin is glued to a glass sheet and the arrangement is held in front of a laser beam, the projected shadow has diffraction rings around its edge and a bright spot in the center. How is this possible?
12. If a fine wire is stretched across the path of a laser beam, is it possible to produce a diffraction pattern?
13. How could the index of refraction of a flat piece of dark obsidian glass be determined?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 38.1 Introduction to Diffraction

Section 38.2 Diffraction from Narrow Slits

1. Helium-neon laser light ($\lambda = 632.8$ nm) is sent through a 0.300-mm-wide single slit. What is the width of the central maximum on a screen 1.00 m from the slit?

2. A beam of green light is diffracted by a slit with a width of 0.550 mm. The diffraction pattern forms on a wall 2.06 m beyond the slit. The distance between the positions of zero intensity on both sides of the central bright fringe is 4.10 mm. Calculate the wavelength of the laser light.

WEB **3.** A screen is placed 50.0 cm from a single slit, which is illuminated with 690-nm light. If the distance between the first and third minima in the diffraction pattern is 3.00 mm, what is the width of the slit?

4. Coherent microwaves of wavelength 5.00 cm enter a long, narrow window in a building otherwise essentially opaque to the microwaves. If the window is 36.0 cm wide, what is the distance from the central maximum to the first-order minimum along a wall 6.50 m from the window?

5. Sound with a frequency of 650 Hz from a distant source passes through a doorway 1.10 m wide in a sound-absorbing wall. Find the number and approximate directions of the diffraction-maximum beams radiated into the space beyond.

6. Light with a wavelength of 587.5 nm illuminates a single slit 0.750 mm in width. (a) At what distance from the slit should a screen be located if the first minimum in the diffraction pattern is to be 0.850 mm from the center of the screen? (b) What is the width of the central maximum?

7. A diffraction pattern is formed on a screen 120 cm away from a 0.400-mm-wide slit. Monochromatic 546.1-nm light is used. Calculate the fractional intensity I/I_0 at a point on the screen 4.10 mm from the center of the principal maximum.

8. The second-order bright fringe in a single-slit diffraction pattern is 1.40 mm from the center of the central maximum. The screen is 80.0 cm from a slit of width 0.800 mm. Assuming that the incident light is monochromatic, calculate the light's approximate wavelength.

9. If the light in Figure 38.5 strikes the single slit at an angle β from the perpendicular direction, show that Equation 38.1, the condition for destructive interference, must be modified to read

$$\sin\theta = m\left(\frac{\lambda}{a}\right) - \sin\beta$$

10. Coherent light with a wavelength of 501.5 nm is sent through two parallel slits in a large flat wall. Each slit is 0.700 μm wide, and the slits' centers are 2.80 μm apart. The light falls on a semicylindrical screen, with its axis at the midline between the slits. (a) Predict the direction of each interference maximum on the screen, as an angle away from the bisector of the line joining the slits. (b) Describe the pattern of light on the screen, specifying the number of bright fringes and the location of each. (c) Find the intensity of light on the screen at the center of each bright fringe, expressed as a fraction of the light intensity I_0 at the center of the pattern.

Section 38.3 Resolution of Single-Slit and Circular Apertures

11. The pupil of a cat's eye narrows to a vertical slit of width 0.500 mm in daylight. What is the angular resolution for horizontally separated mice? Assume that the average wavelength of the light is 500 nm.

12. Find the radius of a star image formed on the retina of the eye if the aperture diameter (the pupil) at night is 0.700 cm and the length of the eye is 3.00 cm. Assume that the representative wavelength of starlight in the eye is 500 nm.

WEB **13.** A helium-neon laser emits light that has a wavelength of 632.8 nm. The circular aperture through which the beam emerges has a diameter of 0.500 cm. Estimate the diameter of the beam 10.0 km from the laser.

14. On the night of April 18, 1775, a signal was to be sent from the steeple of Old North Church in Boston to Paul Revere, who was 1.80 mi away: "One if by land, two if by sea." At what minimum separation did the sexton have to set the lanterns for Revere to receive the correct message? Assume that Revere's pupils had a diameter of 4.00 mm at night and that the lantern light had a predominant wavelength of 580 nm.

15. The Impressionist painter Georges Seurat created paintings with an enormous number of dots of pure pigment, each of which was approximately 2.00 mm in diameter. The idea was to locate colors such as red and green next to each other to form a scintillating canvas (Fig. P38.15). Outside what distance would one be unable to discern individual dots on the canvas? (Assume that $\lambda = 500$ nm and that the pupil diameter is 4.00 mm.)

16. A binary star system in the constellation Orion has an angular interstellar separation of 1.00×10^{-5} rad. If $\lambda = 500$ nm, what is the smallest diameter a telescope must have to just resolve the two stars?

Figure P38.15 *Sunday Afternoon on the Isle of La Grande Jatte,* by Georges Seurat. *(SuperStock)*

17. A child is standing at the edge of a straight highway watching her grandparents' car driving away at 20.0 m/s. The air is perfectly clear and steady, and after 10.0 min the car's two taillights appear to merge into one. Assuming the diameter of the child's pupils is 5.00 mm, estimate the width of the car.

18. Suppose that you are standing on a straight highway and watching a car moving away from you at a speed v. The air is perfectly clear and steady, and after a time t the taillights appear to merge into one. Assuming the diameter of your pupil is d, estimate the width of the car.

19. A circular radar antenna on a Coast Guard ship has a diameter of 2.10 m and radiates at a frequency of 15.0 GHz. Two small boats are located 9.00 km away from the ship. How close together could the boats be and still be detected as two objects?

20. If we were to send a ruby laser beam ($\lambda = 694.3$ nm) outward from the barrel of a 2.70-m-diameter telescope, what would be the diameter of the big red spot when the beam hit the Moon 384 000 km away? (Neglect atmospheric dispersion.)

21. The angular resolution of a radio telescope is to be 0.100° when the incident waves have a wavelength of 3.00 mm. What minimum diameter is required for the telescope's receiving dish?

22. When Mars is nearest the Earth, the distance separating the two planets is 88.6×10^6 km. Mars is viewed through a telescope whose mirror has a diameter of 30.0 cm. (a) If the wavelength of the light is 590 nm, what is the angular resolution of the telescope? (b) What is the smallest distance that can be resolved between two points on Mars?

Section 38.4 The Diffraction Grating

Note: In the following problems, assume that the light is incident normally on the gratings.

23. White light is spread out into its spectral components by a diffraction grating. If the grating has 2 000 lines per centimeter, at what angle does red light of wavelength 640 nm appear in first order?

24. Light from an argon laser strikes a diffraction grating that has 5 310 lines per centimeter. The central and first-order principal maxima are separated by 0.488 m on a wall 1.72 m from the grating. Determine the wavelength of the laser light.

WEB **25.** The hydrogen spectrum has a red line at 656 nm and a violet line at 434 nm. What is the angular separation between two spectral lines obtained with a diffraction grating that has 4 500 lines per centimeter?

26. A helium-neon laser ($\lambda = 632.8$ nm) is used to calibrate a diffraction grating. If the first-order maximum occurs at 20.5°, what is the spacing between adjacent grooves in the grating?

27. Three discrete spectral lines occur at angles of 10.09°, 13.71°, and 14.77° in the first-order spectrum of a grating spectroscope. (a) If the grating has 3 660 slits per centimeter, what are the wavelengths of the light? (b) At what angles are these lines found in the second-order spectrum?

28. A diffraction grating has 800 rulings per millimeter. A beam of light containing wavelengths from 500 to 700 nm hits the grating. Do the spectra of different orders overlap? Explain.

WEB **29.** A diffraction grating with a width of 4.00 cm has been ruled with 3 000 grooves per centimeter. (a) What is the resolving power of this grating in the first three orders? (b) If two monochromatic waves incident on this grating have a mean wavelength of 400 nm, what is their wavelength separation if they are just resolved in the third order?

30. Show that, whenever white light is passed through a diffraction grating of any spacing size, the violet end of the continuous visible spectrum in third order always overlaps the red light at the other end of the second-order spectrum.

31. A source emits 531.62-nm and 531.81-nm light. (a) What minimum number of lines is required for a grating that resolves the two wavelengths in the first-order spectrum? (b) Determine the slit spacing for a grating 1.32 cm wide that has the required minimum number of lines.

32. Two wavelengths λ and $\lambda + \Delta\lambda$ (with $\Delta\lambda \ll \lambda$) are incident on a diffraction grating. Show that the angular separation between the spectral lines in the mth order spectrum is

$$\Delta\theta = \frac{\Delta\lambda}{\sqrt{(d/m)^2 - \lambda^2}}$$

where d is the slit spacing and m is the order number.

33. A grating with 250 lines per millimeter is used with an incandescent light source. Assume that the visible spectrum ranges in wavelength from 400 to 700 nm. In how

many orders can one see (a) the entire visible spectrum and (b) the short-wavelength region?

34. A diffraction grating has 4 200 rulings per centimeter. On a screen 2.00 m from the grating, it is found that for a particular order m, the maxima corresponding to two closely spaced wavelengths of sodium (589.0 nm and 589.6 nm) are separated by 1.59 mm. Determine the value of m.

(Optional)
Section 38.5 Diffraction of X-Rays by Crystals

35. Potassium iodide (KI) has the same crystalline structure as NaCl, with $d = 0.353$ nm. A monochromatic x-ray beam shows a diffraction maximum when the grazing angle is 7.60°. Calculate the x-ray wavelength. (Assume first order.)

36. A wavelength of 0.129 nm characterizes K_α x-rays from zinc. When a beam of these x-rays is incident on the surface of a crystal whose structure is similar to that of NaCl, a first-order maximum is observed at 8.15°. Calculate the interplanar spacing on the basis of this information.

WEB **37.** If the interplanar spacing of NaCl is 0.281 nm, what is the predicted angle at which 0.140-nm x-rays are diffracted in a first-order maximum?

38. The first-order diffraction maximum is observed at 12.6° for a crystal in which the interplanar spacing is 0.240 nm. How many other orders can be observed?

39. Monochromatic x-rays of the K_α line from a nickel target ($\lambda = 0.166$ nm) are incident on a potassium chloride (KCl) crystal surface. The interplanar distance in KCl is 0.314 nm. At what angle (relative to the surface) should the beam be directed for a second-order maximum to be observed?

40. In water of uniform depth, a wide pier is supported on pilings in several parallel rows 2.80 m apart. Ocean waves of uniform wavelength roll in, moving in a direction that makes an angle of 80.0° with the rows of posts. Find the three longest wavelengths of waves that will be strongly reflected by the pilings.

Section 38.6 Polarization of Light Waves

41. Unpolarized light passes through two polaroid sheets. The axis of the first is vertical, and that of the second is at 30.0° to the vertical. What fraction of the initial light is transmitted?

42. Three polarizing disks whose planes are parallel are centered on a common axis. The direction of the transmission axis in each case is shown in Figure P38.42 relative to the common vertical direction. A plane-polarized beam of light with E_0 parallel to the vertical reference direction is incident from the left on the first disk with an intensity of $I_i = 10.0$ units (arbitrary). Calculate the transmitted intensity I_f when (a) $\theta_1 = 20.0°$, $\theta_2 = 40.0°$, and $\theta_3 = 60.0°$; (b) $\theta_1 = 0°$, $\theta_2 = 30.0°$, and $\theta_3 = 60.0°$.

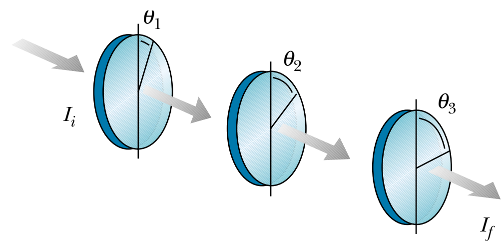


Figure P38.42 Problems 42 and 48.

43. Plane-polarized light is incident on a single polarizing disk with the direction of E_0 parallel to the direction of the transmission axis. Through what angle should the disk be rotated so that the intensity in the transmitted beam is reduced by a factor of (a) 3.00, (b) 5.00, (c) 10.0?

44. The angle of incidence of a light beam onto a reflecting surface is continuously variable. The reflected ray is found to be completely polarized when the angle of incidence is 48.0°. What is the index of refraction of the reflecting material?

45. The critical angle for total internal reflection for sapphire surrounded by air is 34.4°. Calculate the polarizing angle for sapphire.

46. For a particular transparent medium surrounded by air, show that the critical angle for total internal reflection and the polarizing angle are related by the expression $\cot \theta_p = \sin \theta_c$.

47. How far above the horizon is the Moon when its image reflected in calm water is completely polarized? ($n_{\text{water}} = 1.33$.)

ADDITIONAL PROBLEMS

48. In Figure P38.42, suppose that the transmission axes of the left and right polarizing disks are perpendicular to each other. Also, let the center disk be rotated on the common axis with an angular speed ω. Show that if unpolarized light is incident on the left disk with an intensity I_{max}, the intensity of the beam emerging from the right disk is

$$I = \frac{1}{16} I_{\text{max}}(1 - \cos 4\omega t)$$

This means that the intensity of the emerging beam is modulated at a rate that is four times the rate of rotation of the center disk. [*Hint:* Use the trigonometric identities $\cos^2\theta = (1 + \cos 2\theta)/2$ and $\sin^2\theta = (1 - \cos 2\theta)/2$, and recall that $\theta = \omega t$.]

49. You want to rotate the plane of polarization of a polarized light beam by 45.0° with a maximum intensity reduction of 10.0%. (a) How many sheets of perfect polarizers do you need to achieve your goal? (b) What is the angle between adjacent polarizers?

50. Figure P38.50 shows a megaphone in use. Construct a theoretical description of how a megaphone works. You may assume that the sound of your voice radiates just through the opening of your mouth. Most of the information in speech is carried not in a signal at the fundamental frequency, but rather in noises and in harmonics, with frequencies of a few thousand hertz. Does your theory allow any prediction that is simple to test?

Figure P38.50 *(Susan Allen Sigmon/Allsport USA)*

51. Light from a helium-neon laser ($\lambda = 632.8$ nm) is incident on a single slit. What is the maximum width for which no diffraction minima are observed?

52. What are the approximate dimensions of the smallest object on Earth that astronauts can resolve by eye when they are orbiting 250 km above the Earth? Assume that $\lambda = 500$ nm and that a pupil's diameter is 5.00 mm.

53. **Review Problem.** A beam of 541-nm light is incident on a diffraction grating that has 400 lines per millimeter. (a) Determine the angle of the second-order ray. (b) If the entire apparatus is immersed in water, what is the new second-order angle of diffraction? (c) Show that the two diffracted rays of parts (a) and (b) are related through the law of refraction.

54. The Very Large Array is a set of 27 radio telescope dishes in Caton and Socorro Counties, New Mexico (Fig. P38.54). The antennas can be moved apart on railroad tracks, and their combined signals give the resolving power of a synthetic aperture 36.0 km in diameter. (a) If the detectors are tuned to a frequency of 1.40 GHz, what is the angular resolution of the VLA? (b) Clouds of hydrogen radiate at this frequency. What must be the separation distance for two clouds at the center of the galaxy, 26 000 lightyears away, if they are to be resolved? (c) As the telescope looks up, a circling hawk looks down. For comparison, find the angular resolution of the hawk's eye. Assume that it is most sensitive to green light having a wavelength of 500 nm and that it has a pupil with a diameter of 12.0 mm. (d) A mouse is on the ground 30.0 m below. By what distance must the mouse's whiskers be separated for the hawk to resolve them?

Figure P38.54 A rancher in New Mexico rides past one of the 27 radio telescopes that make up the Very Large Array (VLA). *(© Danny Lehman)*

55. Grote Reber was a pioneer in radio astronomy. He constructed a radio telescope with a 10.0-m diameter receiving dish. What was the telescope's angular resolution for 2.00-m radio waves?

56. A 750-nm light beam hits the flat surface of a certain liquid, and the beam is split into a reflected ray and a refracted ray. If the reflected ray is completely polarized at 36.0°, what is the wavelength of the refracted ray?

$\boxed{57.}$ Light of wavelength 500 nm is incident normally on a diffraction grating. If the third-order maximum of the diffraction pattern is observed at 32.0°, (a) what is the number of rulings per centimeter for the grating? (b) Determine the total number of primary maxima that can be observed in this situation.

58. Light strikes a water surface at the polarizing angle. The part of the beam refracted into the water strikes a submerged glass slab (index of refraction, 1.50), as shown in Figure P38.58. If the light reflected from the upper surface of the slab is completely polarized, what is the angle between the water surface and the glass slab?

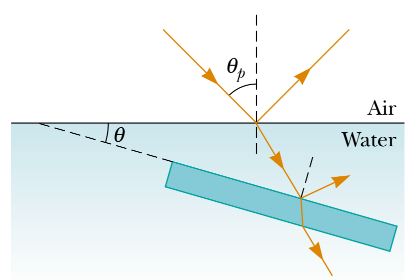


Figure P38.58

59. An American standard television picture is composed of about 485 horizontal lines of varying light intensity. Assume that your ability to resolve the lines is limited only

by the Rayleigh criterion and that the pupils of your eyes are 5.00 mm in diameter. Calculate the ratio of minimum viewing distance to the vertical dimension of the picture such that you will not be able to resolve the lines. Assume that the average wavelength of the light coming from the screen is 550 nm.

60. (a) If light traveling in a medium for which the index of refraction is n_1 is incident at an angle θ on the surface of a medium of index n_2 so that the angle between the reflected and refracted rays is β, show that

$$\tan\theta = \frac{n_2 \sin\beta}{n_1 - n_2\cos\beta}$$

[*Hint:* Use the identity $\sin(A + B) = \sin A\cos B + \cos A \sin B$.] (b) Show that this expression for $\tan\theta$ reduces to Brewster's law when $\beta = 90°$, $n_1 = 1$, and $n_2 = n$.

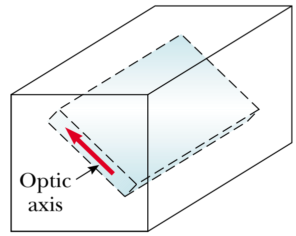
61. Suppose that the single slit in Figure 38.6 is 6.00 cm wide and in front of a microwave source operating at 7.50 GHz. (a) Calculate the angle subtended by the first minimum in the diffraction pattern. (b) What is the relative intensity I/I_{max} at $\theta = 15.0°$? (c) Consider the case when there are two such sources, separated laterally by 20.0 cm, behind the slit. What must the maximum distance between the plane of the sources and the slit be if the diffraction patterns are to be resolved? (In this case, the approximation $\sin\theta \approx \tan\theta$ is not valid because of the relatively small value of a/λ.)

62. Two polarizing sheets are placed together with their transmission axes crossed so that no light is transmitted. A third sheet is inserted between them with its transmission axis at an angle of 45.0° with respect to each of the other axes. Find the fraction of incident unpolarized light intensity transmitted by the three-sheet combination. (Assume that each polarizing sheet is ideal.)

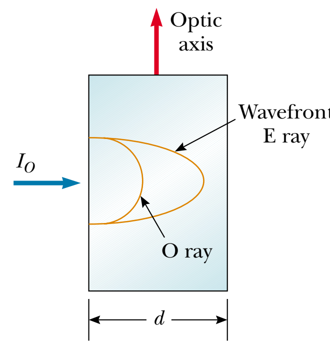
63. Figure P38.63a is a three-dimensional sketch of a birefringent crystal. The dotted lines illustrate how a thin parallel-faced slab of material could be cut from the larger specimen with the optic axis of the crystal parallel to the faces of the plate. A section cut from the crystal in this manner is known as a *retardation plate.* When a beam of light is incident on the plate perpendicular to the direction of the optic axis, as shown in Figure P38.63b, the O ray and the E ray travel along a single straight line but with different speeds. (a) Letting the thickness of the plate be d, show that the phase difference between the O ray and the E ray is

$$\theta = \frac{2\pi d}{\lambda}(n_O - n_E)$$

where λ is the wavelength in air. (b) If in a particular case the incident light has a wavelength of 550 nm, what is the minimum value of d for a quartz plate for which $\theta = \pi/2$? Such a plate is called a *quarter-wave plate.* (Use values of n_O and n_E from Table 38.1.)



(a)



(b)

Figure P38.63

64. Derive Equation 38.12 for the resolving power of a grating, $R = Nm$, where N is the number of lines illuminated and m is the order in the diffraction pattern. Remember that Rayleigh's criterion (see Section 38.3) states that two wavelengths will be resolved when the principal maximum for one falls on the first minimum for the other.

65. Light of wavelength 632.8 nm illuminates a single slit, and a diffraction pattern is formed on a screen 1.00 m from the slit. Using the data in the table on the following page, plot relative intensity versus distance. Choose an appropriate value for the slit width a, and on the same graph used for the experimental data, plot the theoretical expression for the relative intensity

$$\frac{I}{I_{max}} = \frac{\sin^2(\beta/2)}{(\beta/2)^2}$$

What value of a gives the best fit of theory and experiment?

66. How much diffraction spreading does a light beam undergo? One quantitative answer is the *full width at half maximum* of the central maximum of the Fraunhofer diffraction pattern of a single slit. You can evaluate this angle of spreading in this problem and in the next. (a) In Equation 38.4, define $\beta/2 = \phi$ and show that, at the point where $I = 0.5I_{max}$, we must have $\sin\phi = \phi/\sqrt{2}$. (b) Let $y_1 = \sin\phi$ and $y_2 = \phi/\sqrt{2}$. Plot y_1 and y_2 on the same set of axes over a range from $\phi = 1$ rad to $\phi = \pi/2$ rad. Determine ϕ from the point of inter-

Relative Intensity	Distance from Center of Central Maximum (mm)
1.00	0
0.95	0.8
0.80	1.6
0.60	2.4
0.39	3.2
0.21	4.0
0.079	4.8
0.014	5.6
0.003	6.5
0.015	7.3
0.036	8.1
0.047	8.9
0.043	9.7
0.029	10.5
0.013	11.3
0.002	12.1
0.000 3	12.9
0.005	13.7
0.012	14.5
0.016	15.3
0.015	16.1
0.010	16.9
0.004 4	17.7
0.000 6	18.5
0.000 3	19.3
0.003	20.2

section of the two curves. (c) Then show that, if the fraction λ/a is not large, the angular full width at half maximum of the central diffraction maximum is $\Delta\theta = 0.886\lambda/a$.

67. Another method to solve the equation $\phi = \sqrt{2} \sin \phi$ in Problem 66 is to use a calculator, guess a first value of ϕ, see if it fits, and continue to update your estimate until the equation balances. How many steps (iterations) does this take?

68. In the diffraction pattern of a single slit, described by the equation

$$I_\theta = I_{\max}\left[\frac{\sin(\beta/2)}{\beta/2}\right]^2$$

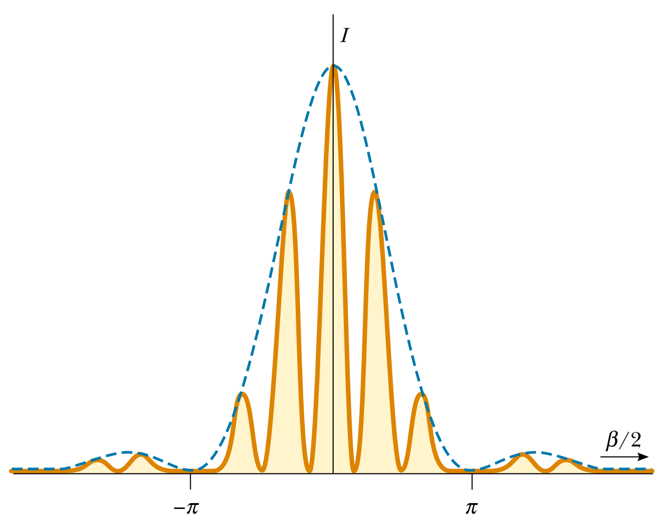
with $\beta = (2\pi a \sin\theta)/\lambda$, the central maximum is at $\beta = 0$ and the side maxima are *approximately* at $\beta/2 = (m + \frac{1}{2})\pi$ for $m = 1, 2, 3, \ldots$. Determine more precisely (a) the location of the first side maximum, where $m = 1$, and (b) the location of the second side maximum. Observe in Figure 38.10a that the graph of intensity versus $\beta/2$ has a horizontal tangent at maxima and also at minima. You will need to solve a transcendental equation.

69. A *pinhole camera* has a small circular aperture of diameter D. Light from distant objects passes through the aperture into an otherwise dark box, falling upon a screen located a distance L away. If D is too large, the display on the screen will be fuzzy because a bright point in the field of view will send light onto a circle of diameter slightly larger than D. On the other hand, if D is too small, diffraction will blur the display on the screen. The screen shows a reasonably sharp image if the diameter of the central disk of the diffraction pattern, specified by Equation 38.9, is equal to D at the screen. (a) Show that for monochromatic light with plane wave fronts and $L \gg D$, the condition for a sharp view is fulfilled if $D^2 = 2.44\,\lambda L$. (b) Find the optimum pinhole diameter if 500-nm light is projected onto a screen 15.0 cm away.

Answers to Quick Quizzes

38.1 The space between the slightly open door and the doorframe acts as a single slit. Sound waves have wavelengths that are approximately the same size as the opening and so are diffracted and spread throughout the room you are in. Because light wavelengths are much smaller than the slit width, they are virtually undiffracted. As a result, you must have a direct line of sight to detect the light waves.

38.2 The situation is like that depicted in Figure 38.11 except that now the slits are only half as far apart. The diffraction pattern is the same, but the interference pattern is stretched out because d is smaller. Because $d/a = 3$, the third interference maximum coincides with the first diffraction minimum. Your sketch should look like the figure to the right.



38.3 Yes, but no diffraction effects are observed because the separation distance between adjacent ribs is so much greater than the wavelength of the x-rays.

PUZZLER

The wristwatches worn by the people in this commercial jetliner properly record the passage of time as experienced by the travelers. Amazingly, however, the duration of the trip as measured by an Earth-bound observer is very slightly longer. How can high-speed travel affect something as regular as the ticking of a clock? *(© Larry Mulvehill/Photo Researchers, Inc.)*

chapter

39 Relativity

Chapter Outline

39.1 The Principle of Galilean Relativity

39.2 The Michelson–Morley Experiment

39.3 Einstein's Principle of Relativity

39.4 Consequences of the Special Theory of Relativity

39.5 The Lorentz Transformation Equations

39.6 Relativistic Linear Momentum and the Relativistic Form of Newton's Laws

39.7 Relativistic Energy

39.8 Equivalence of Mass and Energy

39.9 Relativity and Electromagnetism

39.10 *(Optional)* The General Theory of Relativity

Most of our everyday experiences and observations have to do with objects that move at speeds much less than the speed of light. Newtonian mechanics was formulated to describe the motion of such objects, and this formalism is still very successful in describing a wide range of phenomena that occur at low speeds. It fails, however, when applied to particles whose speeds approach that of light.

Experimentally, the predictions of Newtonian theory can be tested at high speeds by accelerating electrons or other charged particles through a large electric potential difference. For example, it is possible to accelerate an electron to a speed of $0.99c$ (where c is the speed of light) by using a potential difference of several million volts. According to Newtonian mechanics, if the potential difference is increased by a factor of 4, the electron's kinetic energy is four times greater and its speed should double to $1.98c$. However, experiments show that the speed of the electron—as well as the speed of any other particle in the Universe—always remains less than the speed of light, regardless of the size of the accelerating voltage. Because it places no upper limit on speed, Newtonian mechanics is contrary to modern experimental results and is clearly a limited theory.

In 1905, at the age of only 26, Einstein published his special theory of relativity. Regarding the theory, Einstein wrote:

> The relativity theory arose from necessity, from serious and deep contradictions in the old theory from which there seemed no escape. The strength of the new theory lies in the consistency and simplicity with which it solves all these difficulties . . . .[1]

Although Einstein made many other important contributions to science, the special theory of relativity alone represents one of the greatest intellectual achievements of all time. With this theory, experimental observations can be correctly predicted over the range of speeds from $v = 0$ to speeds approaching the speed of light. At low speeds, Einstein's theory reduces to Newtonian mechanics as a limiting situation. It is important to recognize that Einstein was working on electromagnetism when he developed the special theory of relativity. He was convinced that Maxwell's equations were correct, and in order to reconcile them with one of his postulates, he was forced into the bizarre notion of assuming that space and time are not absolute.

This chapter gives an introduction to the special theory of relativity, with emphasis on some of its consequences. The special theory covers phenomena such as the slowing down of clocks and the contraction of lengths in moving reference frames as measured by a stationary observer. We also discuss the relativistic forms of momentum and energy, as well as some consequences of the famous mass–energy formula, $E = mc^2$.

In addition to its well-known and essential role in theoretical physics, the special theory of relativity has practical applications, including the design of nuclear power plants and modern global positioning system (GPS) units. These devices do not work if designed in accordance with nonrelativistic principles.

We shall have occasion to use relativity in some subsequent chapters of the extended version of this text, most often presenting only the outcome of relativistic effects.

[1] A. Einstein and L. Infeld, *The Evolution of Physics,* New York, Simon and Schuster, 1961.

39.1 THE PRINCIPLE OF GALILEAN RELATIVITY

Inertial frame of reference

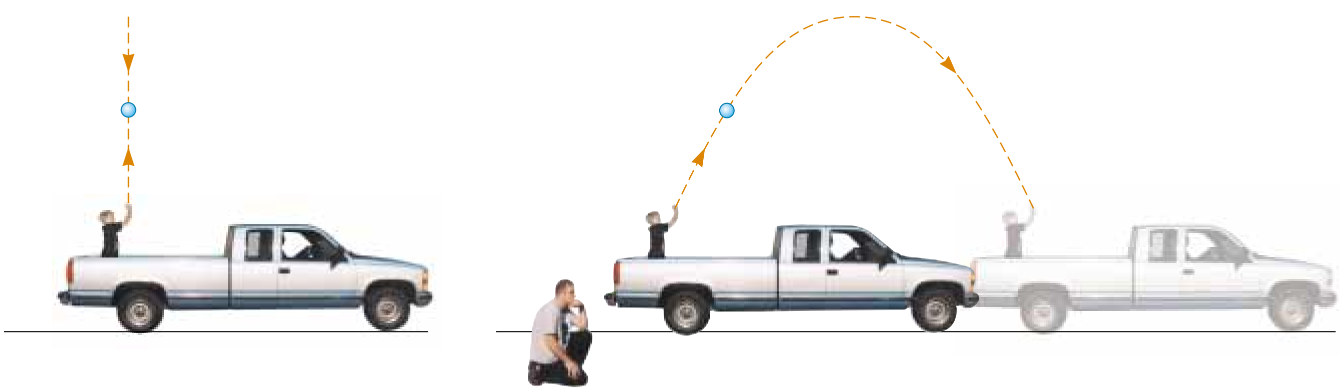
To describe a physical event, it is necessary to establish a frame of reference. You should recall from Chapter 5 that Newton's laws are valid in all inertial frames of reference. Because an inertial frame is defined as one in which Newton's first law is valid, we can say that **an inertial frame of reference is one in which an object is observed to have no acceleration when no forces act on it.** Furthermore, any system moving with constant velocity with respect to an inertial system must also be an inertial system.

There is no preferred inertial reference frame. This means that the results of an experiment performed in a vehicle moving with uniform velocity will be identical to the results of the same experiment performed in a stationary vehicle. The formal statement of this result is called the **principle of Galilean relativity:**

The laws of mechanics must be the same in all inertial frames of reference.

Let us consider an observation that illustrates the equivalence of the laws of mechanics in different inertial frames. A pickup truck moves with a constant velocity, as shown in Figure 39.1a. If a passenger in the truck throws a ball straight up, and if air effects are neglected, the passenger observes that the ball moves in a vertical path. The motion of the ball appears to be precisely the same as if the ball were thrown by a person at rest on the Earth. The law of gravity and the equations of motion under constant acceleration are obeyed whether the truck is at rest or in uniform motion.

Now consider the same situation viewed by an observer at rest on the Earth. This stationary observer sees the path of the ball as a parabola, as illustrated in Figure 39.1b. Furthermore, according to this observer, the ball has a horizontal component of velocity equal to the velocity of the truck. Although the two observers disagree on certain aspects of the situation, they agree on the validity of Newton's laws and on such classical principles as conservation of energy and conservation of linear momentum. This agreement implies that no mechanical experiment can detect any difference between the two inertial frames. The only thing that can be detected is the relative motion of one frame with respect to the other. That is, the notion of absolute motion through space is meaningless, as is the notion of a preferred reference frame.



(a) (b)

Figure 39.1 (a) The observer in the truck sees the ball move in a vertical path when thrown upward. (b) The Earth observer sees the path of the ball as a parabola.

Quick Quiz 39.1

Which observer in Figure 39.1 is right about the ball's path?

Suppose that some physical phenomenon, which we call an *event,* occurs in an inertial system. The event's location and time of occurrence can be specified by the four coordinates (x, y, z, t). We would like to be able to transform these coordinates from one inertial system to another one moving with uniform relative velocity.

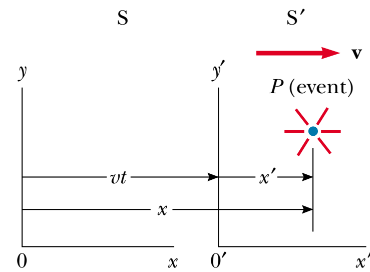


Figure 39.2 An event occurs at a point P. The event is seen by two observers in inertial frames S and S′, where S′ moves with a velocity **v** relative to S.

Consider two inertial systems S and S′ (Fig. 39.2). The system S′ moves with a constant velocity **v** along the xx' axes, where **v** is measured relative to S. We assume that an event occurs at the point P and that the origins of S and S′ coincide at $t = 0$. An observer in S describes the event with space–time coordinates (x, y, z, t), whereas an observer in S′ uses the coordinates (x', y', z', t') to describe the same event. As we see from Figure 39.2, the relationships between these various coordinates can be written

Galilean space–time transformation equations

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{39.1}$$

These equations are the **Galilean space–time transformation equations.** Note that time is assumed to be the same in both inertial systems. That is, within the framework of classical mechanics, all clocks run at the same rate, regardless of their velocity, so that the time at which an event occurs for an observer in S is the same as the time for the same event in S′. Consequently, the time interval between two successive events should be the same for both observers. Although this assumption may seem obvious, it turns out to be incorrect in situations where v is comparable to the speed of light.

Now suppose that a particle moves a distance dx in a time interval dt as measured by an observer in S. It follows from Equations 39.1 that the corresponding distance dx' measured by an observer in S′ is $dx' = dx - v\,dt$, where frame S′ is moving with speed v relative to frame S. Because $dt = dt'$, we find that

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

or

Galilean velocity transformation equation

$$u'_x = u_x - v \tag{39.2}$$

where u_x and u'_x are the x components of the velocity relative to S and S′, respectively. (We use the symbol **u** for particle velocity rather than **v**, which is used for the relative velocity of two reference frames.) This is the **Galilean velocity transformation equation.** It is used in everyday observations and is consistent with our intuitive notion of time and space. As we shall soon see, however, it leads to serious contradictions when applied to electromagnetic waves.

Quick Quiz 39.2

Applying the Galilean velocity transformation equation, determine how fast (relative to the Earth) a baseball pitcher with a 90-mi/h fastball can throw a ball while standing in a boxcar moving at 110 mi/h.

The Speed of Light

It is quite natural to ask whether the principle of Galilean relativity also applies to electricity, magnetism, and optics. Experiments indicate that the answer is no. Recall from Chapter 34 that Maxwell showed that the speed of light in free space is $c = 3.00 \times 10^8$ m/s. Physicists of the late 1800s thought that light waves moved through a medium called the *ether* and that the speed of light was c only in a special, absolute frame at rest with respect to the ether. The Galilean velocity transformation equation was expected to hold in any frame moving at speed v relative to the absolute ether frame.

Because the existence of a preferred, absolute ether frame would show that light was similar to other classical waves and that Newtonian ideas of an absolute frame were true, considerable importance was attached to establishing the existence of the ether frame. Prior to the late 1800s, experiments involving light traveling in media moving at the highest laboratory speeds attainable at that time were not capable of detecting changes as small as $c \pm v$. Starting in about 1880, scientists decided to use the Earth as the moving frame in an attempt to improve their chances of detecting these small changes in the speed of light.

As observers fixed on the Earth, we can say that we are stationary and that the absolute ether frame containing the medium for light propagation moves past us with speed v. Determining the speed of light under these circumstances is just like determining the speed of an aircraft traveling in a moving air current, or wind; consequently, we speak of an "ether wind" blowing through our apparatus fixed to the Earth.

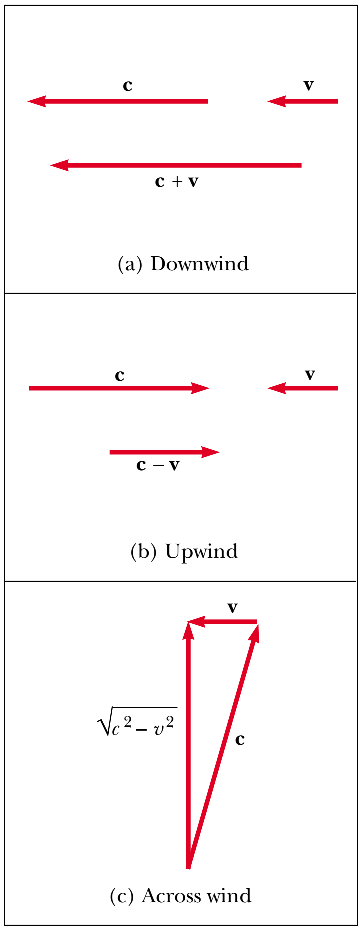


Figure 39.3 If the velocity of the ether wind relative to the Earth is **v** and the velocity of light relative to the ether is **c**, then the speed of light relative to the Earth is (a) $c + v$ in the downwind direction, (b) $c - v$ in the upwind direction, and (c) $(c^2 - v^2)^{1/2}$ in the direction perpendicular to the wind.

A direct method for detecting an ether wind would use an apparatus fixed to the Earth to measure the wind's influence on the speed of light. If v is the speed of the ether relative to the Earth, then the speed of light should have its maximum value, $c + v$, when propagating downwind, as shown in Figure 39.3a. Likewise, the speed of light should have its minimum value, $c - v$, when propagating upwind, as shown in Figure 39.3b, and an intermediate value, $(c^2 - v^2)^{1/2}$, in the direction perpendicular to the ether wind, as shown in Figure 39.3c. If the Sun is assumed to be at rest in the ether, then the velocity of the ether wind would be equal to the orbital velocity of the Earth around the Sun, which has a magnitude of approximately 3×10^4 m/s. Because $c = 3 \times 10^8$ m/s, it should be possible to detect a change in speed of about 1 part in 10^4 for measurements in the upwind or downwind directions. However, as we shall see in the next section, all attempts to detect such changes and establish the existence of the ether wind (and hence the absolute frame) proved futile! (You may want to return to Problem 40 in Chapter 4 to see a situation in which the Galilean velocity transformation equation does hold.)

If it is assumed that the laws of electricity and magnetism are the same in all inertial frames, a paradox concerning the speed of light immediately arises. We can understand this by recognizing that Maxwell's equations seem to imply that the speed of light always has the fixed value 3.00×10^8 m/s in all inertial frames, a result in direct contradiction to what is expected based on the Galilean velocity transformation equation. According to Galilean relativity, the speed of light should not be the same in all inertial frames.

For example, suppose a light pulse is sent out by an observer S′ standing in a boxcar moving with a velocity **v** relative to a stationary observer standing alongside the track (Fig. 39.4). The light pulse has a speed c relative to S′. According to Galilean relativity, the pulse speed relative to S should be $c + v$. This is in contradiction to Einstein's special theory of relativity, which, as we shall see, postulates that the speed of the pulse is the same for all observers.

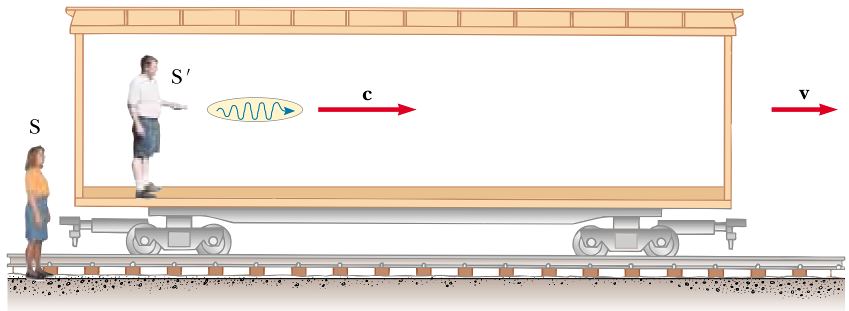


Figure 39.4 A pulse of light is sent out by a person in a moving boxcar. According to Galilean relativity, the speed of the pulse should be $c + v$ relative to a stationary observer.

To resolve this contradiction in theories, we must conclude that either (1) the laws of electricity and magnetism are not the same in all inertial frames or (2) the Galilean velocity transformation equation is incorrect. If we assume the first alternative, then a preferred reference frame in which the speed of light has the value c must exist and the measured speed must be greater or less than this value in any other reference frame, in accordance with the Galilean velocity transformation equation. If we assume the second alternative, then we are forced to abandon the notions of absolute time and absolute length that form the basis of the Galilean space–time transformation equations.

39.2 THE MICHELSON–MORLEY EXPERIMENT

The most famous experiment designed to detect small changes in the speed of light was first performed in 1881 by Albert A. Michelson (see Section 37.7) and later repeated under various conditions by Michelson and Edward W. Morley (1838–1923). We state at the outset that the outcome of the experiment contradicted the ether hypothesis.

The experiment was designed to determine the velocity of the Earth relative to that of the hypothetical ether. The experimental tool used was the Michelson interferometer, which was discussed in Section 37.7 and is shown again in Figure 39.5. Arm 2 is aligned along the direction of the Earth's motion through space. The Earth moving through the ether at speed v is equivalent to the ether flowing past the Earth in the opposite direction with speed v. This ether wind blowing in the direction opposite the direction of Earth's motion should cause the speed of light measured in the Earth frame to be $c - v$ as the light approaches mirror M_2 and $c + v$ after reflection, where c is the speed of light in the ether frame.

The two beams reflected from M_1 and M_2 recombine, and an interference pattern consisting of alternating dark and bright fringes is formed. The interference pattern was observed while the interferometer was rotated through an angle of 90°. This rotation supposedly would change the speed of the ether wind along the arms of the interferometer. The rotation should have caused the fringe pattern to shift slightly but measurably, but measurements failed to show any change in the interference pattern! The Michelson–Morley experiment was repeated at different times of the year when the ether wind was expected to change direction

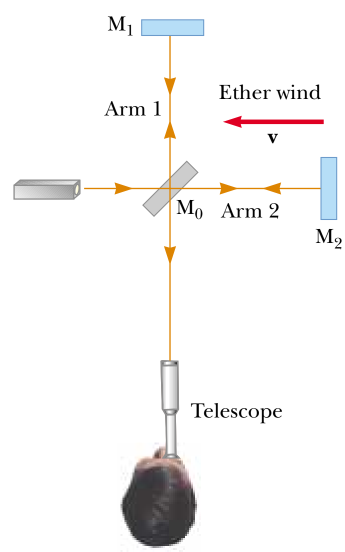
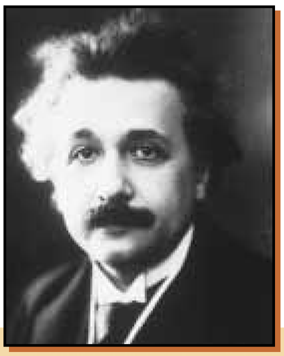


Figure 39.5 According to the ether wind theory, the speed of light should be $c - v$ as the beam approaches mirror M_2 and $c + v$ after reflection.

and magnitude, but the results were always the same: **no fringe shift of the magnitude required was ever observed.**[2]

The negative results of the Michelson–Morley experiment not only contradicted the ether hypothesis but also showed that it was impossible to measure the absolute velocity of the Earth with respect to the ether frame. However, as we shall see in the next section, Einstein offered a postulate for his special theory of relativity that places quite a different interpretation on these null results. In later years, when more was known about the nature of light, the idea of an ether that permeates all of space was relegated to the ash heap of worn-out concepts. **Light is now understood to be an electromagnetic wave, which requires no medium for its propagation.** As a result, the idea of an ether in which these waves could travel became unnecessary.



Albert Einstein (1879–1955) Einstein, one of the greatest physicists of all times, was born in Ulm, Germany. In 1905, at the age of 26, he published four scientific papers that revolutionized physics. Two of these papers were concerned with what is now considered his most important contribution: the special theory of relativity.

In 1916, Einstein published his work on the general theory of relativity. The most dramatic prediction of this theory is the degree to which light is deflected by a gravitational field. Measurements made by astronomers on bright stars in the vicinity of the eclipsed Sun in 1919 confirmed Einstein's prediction, and as a result Einstein became a world celebrity.

Einstein was deeply disturbed by the development of quantum mechanics in the 1920s despite his own role as a scientific revolutionary. In particular, he could never accept the probabilistic view of events in nature that is a central feature of quantum theory. The last few decades of his life were devoted to an unsuccessful search for a unified theory that would combine gravitation and electromagnetism. *(AIP Niels Bohr Library)*

Optional Section

Details of the Michelson–Morley Experiment

To understand the outcome of the Michelson–Morley experiment, let us assume that the two arms of the interferometer in Figure 39.5 are of equal length L. We shall analyze the situation as if there were an ether wind, because that is what Michelson and Morley expected to find. As noted above, the speed of the light beam along arm 2 should be $c - v$ as the beam approaches M_2 and $c + v$ after the beam is reflected. Thus, the time of travel to the right is $L/(c - v)$, and the time of travel to the left is $L/(c + v)$. The total time needed for the round trip along arm 2 is

$$t_1 = \frac{L}{c+v} + \frac{L}{c-v} = \frac{2Lc}{c^2 - v^2} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1}$$

Now consider the light beam traveling along arm 1, perpendicular to the ether wind. Because the speed of the beam relative to the Earth is $(c^2 - v^2)^{1/2}$ in this case (see Fig. 39.3), the time of travel for each half of the trip is $L/(c^2 - v^2)^{1/2}$, and the total time of travel for the round trip is

$$t_2 = \frac{2L}{(c^2 - v^2)^{1/2}} = \frac{2L}{c}\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Thus, the time difference between the horizontal round trip (arm 2) and the vertical round trip (arm 1) is

$$\Delta t = t_1 - t_2 = \frac{2L}{c}\left[\left(1 - \frac{v^2}{c^2}\right)^{-1} - \left(1 - \frac{v^2}{c^2}\right)^{-1/2}\right]$$

Because $v^2/c^2 \ll 1$, we can simplify this expression by using the following binomial expansion after dropping all terms higher than second order:

$$(1 - x)^n \approx 1 - nx \qquad \text{for } x \ll 1$$

In our case, $x = v^2/c^2$, and we find that

$$\Delta t = t_1 - t_2 \approx \frac{Lv^2}{c^3} \qquad \textbf{(39.3)}$$

This time difference between the two instants at which the reflected beams arrive at the viewing telescope gives rise to a phase difference between the beams,

[2] From an Earth observer's point of view, changes in the Earth's speed and direction of motion in the course of a year are viewed as ether wind shifts. Even if the speed of the Earth with respect to the ether were zero at some time, six months later the speed of the Earth would be 60 km/s with respect to the ether, and as a result a fringe shift should be noticed. No shift has ever been observed, however.

producing an interference pattern when they combine at the position of the telescope. A shift in the interference pattern should be detected when the interferometer is rotated through 90° in a horizontal plane, so that the two beams exchange roles. This results in a time difference twice that given by Equation 39.3. Thus, the path difference that corresponds to this time difference is

$$\Delta d = c(2\,\Delta t) = \frac{2Lv^2}{c^2}$$

Because a change in path length of one wavelength corresponds to a shift of one fringe, the corresponding fringe shift is equal to this path difference divided by the wavelength of the light:

$$\text{Shift} = \frac{2Lv^2}{\lambda c^2} \tag{39.4}$$

In the experiments by Michelson and Morley, each light beam was reflected by mirrors many times to give an effective path length L of approximately 11 m. Using this value and taking v to be equal to 3.0×10^4 m/s, the speed of the Earth around the Sun, we obtain a path difference of

$$\Delta d = \frac{2(11\text{ m})(3.0 \times 10^4\text{ m/s})^2}{(3.0 \times 10^8\text{ m/s})^2} = 2.2 \times 10^{-7}\text{ m}$$

This extra travel distance should produce a noticeable shift in the fringe pattern. Specifically, using 500-nm light, we expect a fringe shift for rotation through 90° of

$$\text{Shift} = \frac{\Delta d}{\lambda} = \frac{2.2 \times 10^{-7}\text{ m}}{5.0 \times 10^{-7}\text{ m}} \approx 0.44$$

The instrument used by Michelson and Morley could detect shifts as small as 0.01 fringe. However, **it detected no shift whatsoever in the fringe pattern.** Since then, the experiment has been repeated many times by different scientists under a wide variety of conditions, and no fringe shift has ever been detected. Thus, it was concluded that the motion of the Earth with respect to the postulated ether cannot be detected.

Many efforts were made to explain the null results of the Michelson–Morley experiment and to save the ether frame concept and the Galilean velocity transformation equation for light. All proposals resulting from these efforts have been shown to be wrong. No experiment in the history of physics received such valiant efforts to explain the absence of an expected result as did the Michelson–Morley experiment. The stage was set for Einstein, who solved the problem in 1905 with his special theory of relativity.

39.3 EINSTEIN'S PRINCIPLE OF RELATIVITY

In the previous section we noted the impossibility of measuring the speed of the ether with respect to the Earth and the failure of the Galilean velocity transformation equation in the case of light. Einstein proposed a theory that boldly removed these difficulties and at the same time completely altered our notion of space and time.[3] He based his special theory of relativity on two postulates:

[3] A. Einstein, "On the Electrodynamics of Moving Bodies," *Ann. Physik* 17:891, 1905. For an English translation of this article and other publications by Einstein, see the book by H. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, *The Principle of Relativity,* Dover, 1958.

The postulates of the special theory of relativity

1. **The principle of relativity:** The laws of physics must be the same in all inertial reference frames.
2. **The constancy of the speed of light:** The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

The first postulate asserts that *all* the laws of physics—those dealing with mechanics, electricity and magnetism, optics, thermodynamics, and so on—are the same in all reference frames moving with constant velocity relative to one another. This postulate is a sweeping generalization of the principle of Galilean relativity, which refers only to the laws of mechanics. From an experimental point of view, Einstein's principle of relativity means that any kind of experiment (measuring the speed of light, for example) performed in a laboratory at rest must give the same result when performed in a laboratory moving at a constant velocity past the first one. Hence, no preferred inertial reference frame exists, and it is impossible to detect absolute motion.

Note that postulate 2 is required by postulate 1: If the speed of light were not the same in all inertial frames, measurements of different speeds would make it possible to distinguish between inertial frames; as a result, a preferred, absolute frame could be identified, in contradiction to postulate 1.

Although the Michelson–Morley experiment was performed before Einstein published his work on relativity, it is not clear whether or not Einstein was aware of the details of the experiment. Nonetheless, the null result of the experiment can be readily understood within the framework of Einstein's theory. According to his principle of relativity, the premises of the Michelson–Morley experiment were incorrect. In the process of trying to explain the expected results, we stated that when light traveled against the ether wind its speed was $c - v$, in accordance with the Galilean velocity transformation equation. However, if the state of motion of the observer or of the source has no influence on the value found for the speed of light, one always measures the value to be c. Likewise, the light makes the return trip after reflection from the mirror at speed c, not at speed $c + v$. Thus, the motion of the Earth does not influence the fringe pattern observed in the Michelson–Morley experiment, and a null result should be expected.

If we accept Einstein's theory of relativity, we must conclude that relative motion is unimportant when measuring the speed of light. At the same time, we shall see that we must alter our common-sense notion of space and time and be prepared for some bizarre consequences. It may help as you read the pages ahead to keep in mind that our common-sense ideas are based on a lifetime of everyday experiences and not on observations of objects moving at hundreds of thousands of kilometers per second.

39.4 CONSEQUENCES OF THE SPECIAL THEORY OF RELATIVITY

Before we discuss the consequences of Einstein's special theory of relativity, we must first understand how an observer located in an inertial reference frame describes an event. As mentioned earlier, an event is an occurrence describable by three space coordinates and one time coordinate. Different observers in different inertial frames usually describe the same event with different coordinates.

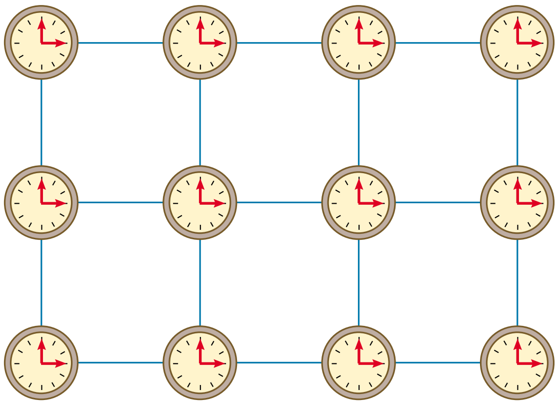


Figure 39.6 In studying relativity, we use a reference frame consisting of a coordinate grid and a set of synchronized clocks.

The reference frame used to describe an event consists of a coordinate grid and a set of synchronized clocks located at the grid intersections, as shown in Figure 39.6 in two dimensions. The clocks can be synchronized in many ways with the help of light signals. For example, suppose an observer is located at the origin with a master clock and sends out a pulse of light at $t = 0$. The pulse takes a time r/c to reach a clock located a distance r from the origin. Hence, this clock is synchronized with the master clock if this clock reads r/c at the instant the pulse reaches it. This procedure of synchronization assumes that the speed of light has the same value in all directions and in all inertial frames. Furthermore, the procedure concerns an event recorded by an observer in a specific inertial reference frame. An observer in some other inertial frame would assign different space–time coordinates to events being observed by using another coordinate grid and another array of clocks.

As we examine some of the consequences of relativity in the remainder of this section, we restrict our discussion to the concepts of simultaneity, time, and length, all three of which are quite different in relativistic mechanics from what they are in Newtonian mechanics. For example, in relativistic mechanics the distance between two points and the time interval between two events depend on the frame of reference in which they are measured. That is, **in relativistic mechanics there is no such thing as absolute length or absolute time.** Furthermore, **events at different locations that are observed to occur simultaneously in one frame are not observed to be simultaneous in another frame moving uniformly past the first.**

Simultaneity and the Relativity of Time

A basic premise of Newtonian mechanics is that a universal time scale exists that is the same for all observers. In fact, Newton wrote that “Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external.” Thus, Newton and his followers simply took simultaneity for granted. In his special theory of relativity, Einstein abandoned this assumption.

Einstein devised the following thought experiment to illustrate this point. A boxcar moves with uniform velocity, and two lightning bolts strike its ends, as illustrated in Figure 39.7a, leaving marks on the boxcar and on the ground. The marks on the boxcar are labeled A' and B', and those on the ground are labeled A and B. An observer O' moving with the boxcar is midway between A' and B', and a ground observer O is midway between A and B. The events recorded by the observers are the striking of the boxcar by the two lightning bolts.

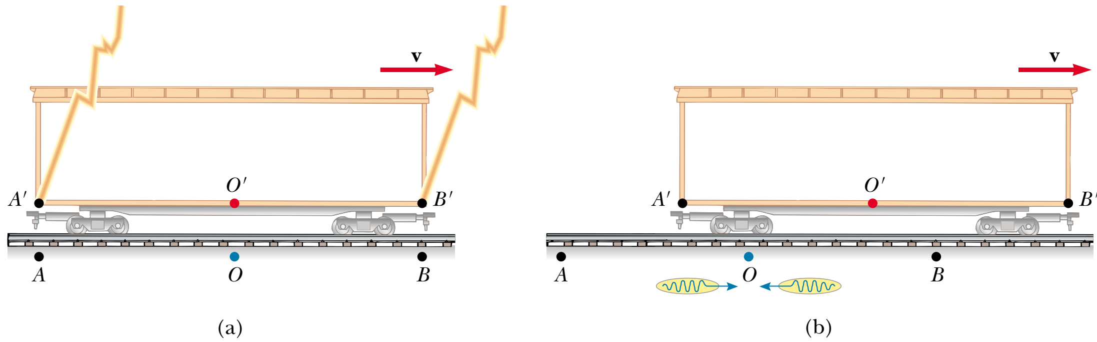


Figure 39.7 (a) Two lightning bolts strike the ends of a moving boxcar. (b) The events appear to be simultaneous to the stationary observer O, standing midway between A and B. The events do not appear to be simultaneous to observer O', who claims that the front of the car is struck before the rear. Note that in (b) the leftward-traveling light signal has already passed O' but the rightward-traveling signal has not yet reached O'.

The light signals recording the instant at which the two bolts strike reach observer O at the same time, as indicated in Figure 39.7b. This observer realizes that the signals have traveled at the same speed over equal distances, and so rightly concludes that the events at A and B occurred simultaneously. Now consider the same events as viewed by observer O'. By the time the signals have reached observer O, observer O' has moved as indicated in Figure 39.7b. Thus, the signal from B' has already swept past O', but the signal from A' has not yet reached O'. In other words, O' sees the signal from B' before seeing the signal from A'. According to Einstein, *the two observers must find that light travels at the same speed.* Therefore, observer O' concludes that the lightning strikes the front of the boxcar before it strikes the back.

This thought experiment clearly demonstrates that the two events that appear to be simultaneous to observer O do not appear to be simultaneous to observer O'. In other words,

two events that are simultaneous in one reference frame are in general not simultaneous in a second frame moving relative to the first. That is, simultaneity is not an absolute concept but rather one that depends on the state of motion of the observer.

Quick Quiz 39.3

Which observer in Figure 39.7 is correct?

The central point of relativity is this: Any inertial frame of reference can be used to describe events and do physics. **There is no preferred inertial frame of reference.** However, observers in different inertial frames always measure different time intervals with their clocks and different distances with their meter sticks. Nevertheless, all observers agree on the forms of the laws of physics in their respective frames because these laws must be the same for all observers in uniform motion. For example, the relationship $F = ma$ in a frame S has the same form $F' = ma'$ in a frame S′ that is moving at constant velocity relative to frame S. It is

the alteration of time and space that allows the laws of physics (including Maxwell's equations) to be the same for all observers in uniform motion.

Time Dilation

We can illustrate the fact that observers in different inertial frames always measure different time intervals between a pair of events by considering a vehicle moving to the right with a speed v, as shown in Figure 39.8a. A mirror is fixed to the ceiling of the vehicle, and observer O' at rest in this system holds a laser a distance d below the mirror. At some instant, the laser emits a pulse of light directed toward the mirror (event 1), and at some later time after reflecting from the mirror, the pulse arrives back at the laser (event 2). Observer O' carries a clock C' and uses it to measure the time interval Δt_p between these two events. (The subscript p stands for *proper,* as we shall see in a moment.) Because the light pulse has a speed c, the time it takes the pulse to travel from O' to the mirror and back to O' is

$$\Delta t_p = \frac{\text{Distance traveled}}{\text{Speed}} = \frac{2d}{c} \qquad \textbf{(39.5)}$$

This time interval Δt_p measured by O' requires only a single clock C' located at the same place as the laser in this frame.

Now consider the same pair of events as viewed by observer O in a second frame, as shown in Figure 39.8b. According to this observer, the mirror and laser are moving to the right with a speed v, and as a result the sequence of events appears entirely different. By the time the light from the laser reaches the mirror, the mirror has moved to the right a distance $v\,\Delta t/2$, where Δt is the time it takes the light to travel from O' to the mirror and back to O' as measured by O. In other words, O concludes that, because of the motion of the vehicle, if the light is to hit the mirror, it must leave the laser at an angle with respect to the vertical direction. Comparing Figure 39.8a and b, we see that the light must travel farther in (b) than in (a). (Note that neither observer "knows" that he or she is moving. Each is at rest in his or her own inertial frame.)

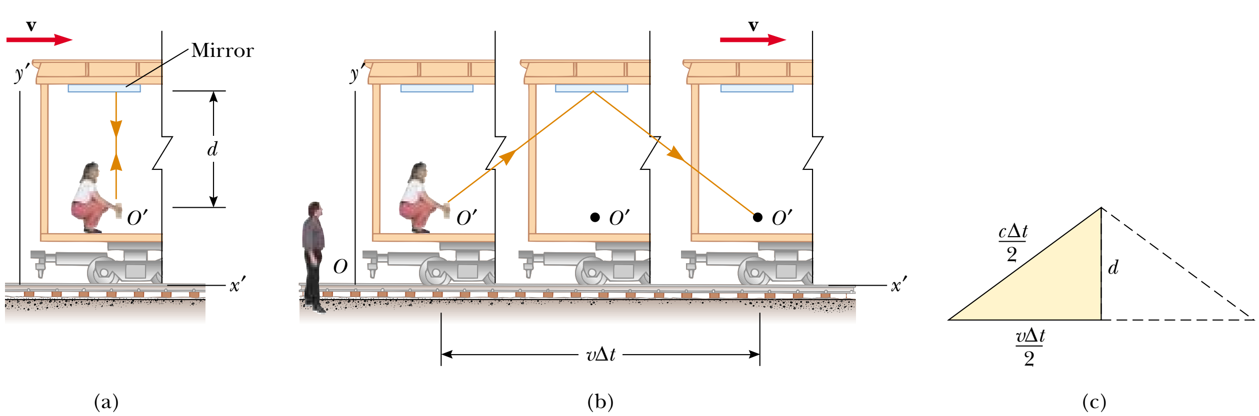


Figure 39.8 (a) A mirror is fixed to a moving vehicle, and a light pulse is sent out by observer O' at rest in the vehicle. (b) Relative to a stationary observer O standing alongside the vehicle, the mirror and O' move with a speed v. Note that what observer O measures for the distance the pulse travels is greater than $2d$. (c) The right triangle for calculating the relationship between Δt and Δt_p.

According to the second postulate of the special theory of relativity, both observers must measure c for the speed of light. Because the light travels farther in the frame of O, it follows that the time interval Δt measured by O is longer than the time interval Δt_p measured by O'. To obtain a relationship between these two time intervals, it is convenient to use the right triangle shown in Figure 39.8c. The Pythagorean theorem gives

$$\left(\frac{c\,\Delta t}{2}\right)^2 = \left(\frac{v\,\Delta t}{2}\right)^2 + d^2$$

Solving for Δt gives

$$\Delta t = \frac{2d}{\sqrt{c^2 - v^2}} = \frac{2d}{c\sqrt{1 - \frac{v^2}{c^2}}} \quad \textbf{(39.6)}$$

Because $\Delta t_p = 2d/c$, we can express this result as

Time dilation

$$\Delta t = \frac{\Delta t_p}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma\,\Delta t_p \quad \textbf{(39.7)}$$

where

$$\gamma = (1 - v^2/c^2)^{-1/2} \quad \textbf{(39.8)}$$

Because γ is always greater than unity, this result says that **the time interval Δt measured by an observer moving with respect to a clock is longer than the time interval Δt_p measured by an observer at rest with respect to the clock.** (That is, $\Delta t > \Delta t_p$.) This effect is known as **time dilation.** Figure 39.9 shows that as the velocity approaches the speed of light, γ increases dramatically. Note that for speeds less than one tenth the speed of light, γ is very nearly equal to unity.

The time interval Δt_p in Equations 39.5 and 39.7 is called the **proper time.** (In German, Einstein used the term *Eigenzeit,* which means "own-time.") In general, **proper time is the time interval between two events measured by an observer who sees the events occur at the same point in space.** Proper time is always the time measured with a single clock (clock C' in our case) at rest in the frame in which the events take place.

If a clock is moving with respect to you, it appears to fall behind (tick more slowly than) the clocks it is passing in the grid of synchronized clocks in your reference frame. Because the time interval $\gamma(2d/c)$, the interval between ticks of a moving

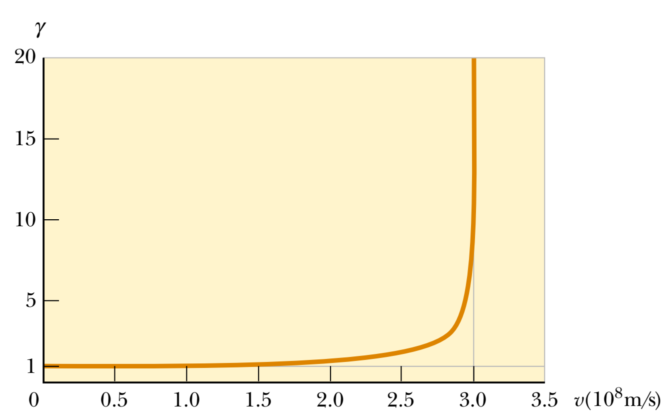


Figure 39.9 Graph of γ versus v. As the velocity approaches the speed of light, γ increases rapidly.

clock, is observed to be longer than $2d/c$, the time interval between ticks of an identical clock in your reference frame, it is often said that a moving clock runs more slowly than a clock in your reference frame by a factor γ. This is true for mechanical clocks as well as for the light clock just described. We can generalize this result by stating that all physical processes, including chemical and biological ones, slow down relative to a stationary clock when those processes occur in a moving frame. For example, the heartbeat of an astronaut moving through space would keep time with a clock inside the spaceship. Both the astronaut's clock and heartbeat would be slowed down relative to a stationary clock back on the Earth (although the astronaut would have no sensation of life slowing down in the spaceship).

Quick Quiz 39.4

A rocket has a clock built into its control panel. Use Figure 39.9 to determine approximately how fast the rocket must be moving before its clock appears to an Earth-bound observer to be ticking at one fifth the rate of a clock on the wall at Mission Control. What does an astronaut in the rocket observe?

Bizarre as it may seem, time dilation is a verifiable phenomenon. An experiment reported by Hafele and Keating provided direct evidence of time dilation.[4] Time intervals measured with four cesium atomic clocks in jet flight were compared with time intervals measured by Earth-based reference atomic clocks. In order to compare these results with theory, many factors had to be considered, including periods of acceleration and deceleration relative to the Earth, variations in direction of travel, and the fact that the gravitational field experienced by the flying clocks was weaker than that experienced by the Earth-based clock. The results were in good agreement with the predictions of the special theory of relativity and can be explained in terms of the relative motion between the Earth and the jet aircraft. In their paper, Hafele and Keating stated that "Relative to the atomic time scale of the U.S. Naval Observatory, the flying clocks lost 59 ± 10 ns during the eastward trip and gained 273 ± 7 ns during the westward trip . . . . These results provide an unambiguous empirical resolution of the famous clock paradox with macroscopic clocks."

Another interesting example of time dilation involves the observation of *muons,* unstable elementary particles that have a charge equal to that of the electron and a mass 207 times that of the electron. Muons can be produced by the collision of cosmic radiation with atoms high in the atmosphere. These particles have a lifetime of 2.2 μs when measured in a reference frame in which they are at rest or moving slowly. If we take 2.2 μs as the average lifetime of a muon and assume that its speed is close to the speed of light, we find that these particles travel only approximately 600 m before they decay (Fig. 39.10a). Hence, they cannot reach the Earth from the upper atmosphere where they are produced. However, experiments show that a large number of muons do reach the Earth. The phenomenon of time dilation explains this effect. Relative to an observer on the Earth, the muons have a lifetime equal to $\gamma\tau_p$, where $\tau_p = 2.2\ \mu$s is the lifetime in the frame traveling with the muons or the proper lifetime. For example, for a muon speed of $v = 0.99c$, $\gamma \approx 7.1$ and $\gamma\tau_p \approx 16\ \mu$s. Hence, the average distance traveled as measured by an observer on the Earth is $\gamma v \tau_p \approx 4\,800$ m, as indicated in Figure 39.10b.

In 1976, at the laboratory of the European Council for Nuclear Research

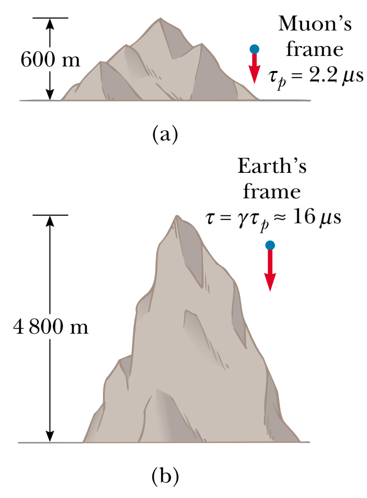


Figure 39.10 (a) Muons moving with a speed of $0.99c$ travel approximately 600 m as measured in the reference frame of the muons, where their lifetime is about 2.2 μs. (b) The muons travel approximately 4 800 m as measured by an observer on the Earth.

[4] J. C. Hafele and R. E. Keating, "Around the World Atomic Clocks: Relativistic Time Gains Observed," *Science,* 177:168, 1972.

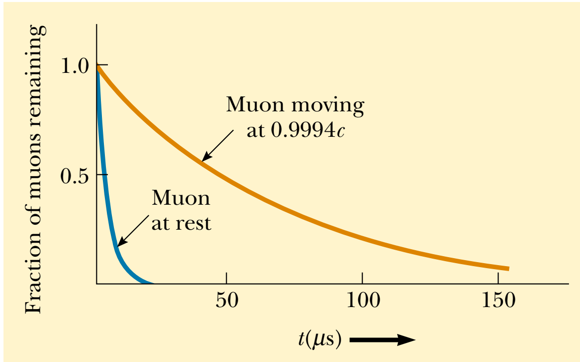


Figure 39.11 Decay curves for muons at rest and for muons traveling at a speed of $0.9994c$.

(CERN) in Geneva, muons injected into a large storage ring reached speeds of approximately $0.9994c$. Electrons produced by the decaying muons were detected by counters around the ring, enabling scientists to measure the decay rate and hence the muon lifetime. The lifetime of the moving muons was measured to be approximately 30 times as long as that of the stationary muon (Fig. 39.11), in agreement

EXAMPLE 39.1 What Is the Period of the Pendulum?

The period of a pendulum is measured to be 3.0 s in the reference frame of the pendulum. What is the period when measured by an observer moving at a speed of $0.95c$ relative to the pendulum?

Solution Instead of the observer moving at $0.95c$, we can take the equivalent point of view that the observer is at rest and the pendulum is moving at $0.95c$ past the stationary observer. Hence, the pendulum is an example of a moving clock.

The proper time is $\Delta t_p = 3.0$ s. Because a moving clock runs more slowly than a stationary clock by a factor γ, Equation 39.7 gives

$$\Delta t = \gamma \Delta t_p = \frac{1}{\sqrt{1 - \frac{(0.95c)^2}{c^2}}} \Delta t_p = \frac{1}{\sqrt{1 - 0.902}} \Delta t_p$$

$$= (3.2)(3.0 \text{ s}) = \boxed{9.6 \text{ s}}$$

That is, a moving pendulum takes longer to complete a period than a pendulum at rest does.

EXAMPLE 39.2 How Long Was Your Trip?

Suppose you are driving your car on a business trip and are traveling at 30 m/s. Your boss, who is waiting at your destination, expects the trip to take 5.0 h. When you arrive late, your excuse is that your car clock registered the passage of 5.0 h but that you were driving fast and so your clock ran more slowly than your boss's clock. If your car clock actually did indicate a 5.0-h trip, how much time passed on your boss's clock, which was at rest on the Earth?

Solution We begin by calculating γ from Equation 39.8:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(3 \times 10^1 \text{ m/s})^2}{(3 \times 10^8 \text{ m/s})^2}}} = \frac{1}{\sqrt{1 - 10^{-14}}}$$

If you try to determine this value on your calculator, you will probably get $\gamma = 1$. However, if we perform a binomial expansion, we can more precisely determine the value as

$$\gamma = (1 - 10^{-14})^{-1/2} \approx 1 + \tfrac{1}{2}(10^{-14}) = 1 + 5.0 \times 10^{-15}$$

This result indicates that at typical automobile speeds, γ is not much different from 1.

Applying Equation 39.7, we find Δt, the time interval measured by your boss, to be

$$\Delta t = \gamma \Delta t_p = (1 + 5.0 \times 10^{-15})(5.0 \text{ h})$$

$$= 5.0 \text{ h} + 2.5 \times 10^{-14} \text{ h} = \boxed{5.0 \text{ h} + 0.09 \text{ ns}}$$

Your boss's clock would be only 0.09 ns ahead of your car clock. You might want to try another excuse!

with the prediction of relativity to within two parts in a thousand.

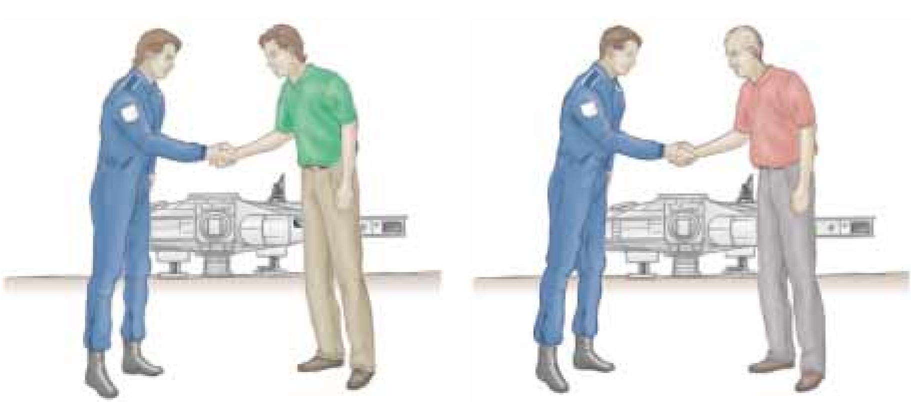
The Twins Paradox

An intriguing consequence of time dilation is the so-called *twins paradox* (Fig. 39.12). Consider an experiment involving a set of twins named Speedo and Goslo. When they are 20 yr old, Speedo, the more adventuresome of the two, sets out on an epic journey to Planet X, located 20 ly from the Earth. Furthermore, his spaceship is capable of reaching a speed of $0.95c$ relative to the inertial frame of his twin brother back home. After reaching Planet X, Speedo becomes homesick and immediately returns to the Earth at the same speed $0.95c$. Upon his return, Speedo is shocked to discover that Goslo has aged 42 yr and is now 62 yr old. Speedo, on the other hand, has aged only 13 yr.

At this point, it is fair to raise the following question—which twin is the traveler and which is really younger as a result of this experiment? From Goslo's frame of reference, he was at rest while his brother traveled at a high speed. But from Speedo's perspective, it is he who was at rest while Goslo was on the high-speed space journey. According to Speedo, he himself remained stationary while Goslo and the Earth raced away from him on a 6.5-yr journey and then headed back for another 6.5 yr. This leads to an apparent contradiction. Which twin has developed signs of excess aging?

To resolve this apparent paradox, recall that the special theory of relativity deals with inertial frames of reference moving relative to each other at uniform speed. However, the trip in our current problem is not symmetrical. Speedo, the space traveler, must experience a series of accelerations during his journey. As a result, his speed is not always uniform, and consequently he is not in an inertial frame. He cannot be regarded as always being at rest while Goslo is in uniform motion because to do so would be an incorrect application of the special theory of relativity. Therefore, there is no paradox. During each passing year noted by Goslo, slightly less than 4 months elapsed for Speedo.

The conclusion that Speedo is in a noninertial frame is inescapable. Each twin observes the other as accelerating, but it is Speedo that actually undergoes dynamical acceleration due to the real forces acting on him. The time required to accelerate and decelerate Speedo's spaceship may be made very small by using large rockets, so that Speedo can claim that he spends most of his time traveling to Planet X



(a) (b)

Figure 39.12 (a) As one twin leaves his brother on the Earth, both are the same age. (b) When Speedo returns from his journey to Planet X, he is younger than his twin Goslo.

at $0.95c$ in an inertial frame. However, Speedo must slow down, reverse his motion, and return to the Earth in an altogether different inertial frame. At the very best, Speedo is in two different inertial frames during his journey. Only Goslo, who is in a single inertial frame, can apply the simple time-dilation formula to Speedo's trip. Thus, Goslo finds that instead of aging 42 yr, Speedo ages only $(1 - v^2/c^2)^{1/2}(42\text{ yr}) = 13\text{ yr}$. Conversely, Speedo spends 6.5 yr traveling to Planet X and 6.5 yr returning, for a total travel time of 13 yr, in agreement with our earlier statement.

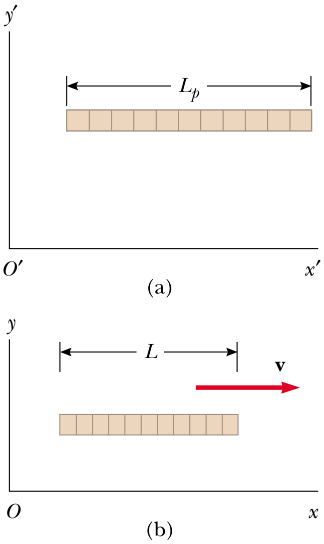


Figure 39.13 (a) A stick measured by an observer in a frame attached to the stick (that is, both have the same velocity) has its proper length L_p. (b) The stick measured by an observer in a frame in which the stick has a velocity **v** relative to the frame is shorter than its proper length L_p by a factor $(1 - v^2/c^2)^{1/2}$.

Quick Quiz 39.5

Suppose astronauts are paid according to the amount of time they spend traveling in space. After a long voyage traveling at a speed approaching c, would a crew rather be paid according to an Earth-based clock or their spaceship's clock?

Length Contraction

The measured distance between two points also depends on the frame of reference. **The proper length L_p of an object is the length measured by someone at rest relative to the object.** The length of an object measured by someone in a reference frame that is moving with respect to the object is always less than the proper length. This effect is known as **length contraction.**

Consider a spaceship traveling with a speed v from one star to another. There are two observers: one on the Earth and the other in the spaceship. The observer at rest on the Earth (and also assumed to be at rest with respect to the two stars) measures the distance between the stars to be the proper length L_p. According to this observer, the time it takes the spaceship to complete the voyage is $\Delta t = L_p/v$. Because of time dilation, the space traveler measures a smaller time of travel by the spaceship clock: $\Delta t_p = \Delta t/\gamma$. The space traveler claims to be at rest and sees the destination star moving toward the spaceship with speed v. Because the space traveler reaches the star in the time Δt_p, he or she concludes that the distance L between the stars is shorter than L_p. This distance measured by the space traveler is

$$L = v\,\Delta t_p = v\,\frac{\Delta t}{\gamma}$$

Because $L_p = v\,\Delta t$, we see that

Length contraction

$$L = \frac{L_p}{\gamma} = L_p\left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{39.9}$$

If an object has a proper length L_p when it is at rest, then when it moves with speed v in a direction parallel to its length, it contracts to the length $L = L_p(1 - v^2/c^2)^{1/2} = L_p/\gamma$.

where $(1 - v^2/c^2)^{1/2}$ is a factor less than unity. This result may be interpreted as follows:

For example, suppose that a stick moves past a stationary Earth observer with speed v, as shown in Figure 39.13. The length of the stick as measured by an observer in a frame attached to the stick is the proper length L_p shown in Figure 39.13a. The length of the stick L measured by the Earth observer is shorter than L_p by the factor $(1 - v^2/c^2)^{1/2}$. Furthermore, length contraction is a symmetrical effect: If the stick is at rest on the Earth, an observer in a moving frame would

measure its length to be shorter by the same factor $(1 - v^2/c^2)^{1/2}$. Note that **length contraction takes place only along the direction of motion.**

It is important to emphasize that proper length and proper time are measured in different reference frames. As an example of this point, let us return to the decaying muons moving at speeds close to the speed of light. An observer in the muon reference frame measures the proper lifetime (that is, the time interval τ_p), whereas an Earth-based observer measures a dilated lifetime. However, the Earth-based observer measures the proper height (the length L_p) of the mountain in Figure 39.10b. In the muon reference frame, this height is less than L_p, as the figure shows. Thus, in the muon frame, length contraction occurs but time dilation does not. In the Earth-based reference frame, time dilation occurs but length contraction does not. Thus, when calculations on the muon are performed in both

EXAMPLE 39.3 The Contraction of a Spaceship

A spaceship is measured to be 120.0 m long and 20.0 m in diameter while at rest relative to an observer. If this spaceship now flies by the observer with a speed of $0.99c$, what length and diameter does the observer measure?

Solution From Equation 39.9, the length measured by the observer is

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}} = (120.0\text{ m})\sqrt{1 - \frac{(0.99c)^2}{c^2}} = \boxed{17\text{ m}}$$

The diameter measured by the observer is still 20.0 m because the diameter is a dimension perpendicular to the motion and length contraction occurs only along the direction of motion.

Exercise If the ship moves past the observer with a speed of $0.100\,0c$, what length does the observer measure?

Answer 119.4 m.

EXAMPLE 39.4 How Long Was Your Car?

In Example 39.2, you were driving at 30 m/s and claimed that your clock was running more slowly than your boss's stationary clock. Although your statement was true, the time dilation was negligible. If your car is 4.3 m long when it is parked, how much shorter does it appear to a stationary roadside observer as you drive by at 30 m/s?

Solution The observer sees the horizontal length of the car to be contracted to a length

$$L = L_p\sqrt{1 - \frac{v^2}{c^2}} \approx L_p\left(1 - \frac{1}{2}\frac{v^2}{c^2}\right)$$

where we have again used the binomial expansion for the factor $\sqrt{1 - \frac{v^2}{c^2}}$. The roadside observer sees the car's length as having changed by an amount $L_p - L$:

$$L_p - L \approx \frac{L_p}{2}\left(\frac{v^2}{c^2}\right) = \left(\frac{4.3\text{ m}}{2}\right)\left(\frac{3.0 \times 10^1\text{ m/s}}{3.0 \times 10^8\text{ m/s}}\right)^2$$

$$= \boxed{2.2 \times 10^{-14}\text{ m}}$$

This is much smaller than the diameter of an atom!

EXAMPLE 39.5 A Voyage to Sirius

An astronaut takes a trip to Sirius, which is located a distance of 8 lightyears from the Earth. (Note that 1 lightyear (ly) is the distance light travels through free space in 1 yr.) The astronaut measures the time of the one-way journey to be 6 yr. If the spaceship moves at a constant speed of $0.8c$, how can the 8-ly distance be reconciled with the 6-yr trip time measured by the astronaut?

Solution The 8 ly represents the proper length from the Earth to Sirius measured by an observer seeing both bodies nearly at rest. The astronaut sees Sirius approaching her at $0.8c$ but also sees the distance contracted to

$$\frac{8\text{ ly}}{\gamma} = (8\text{ ly})\sqrt{1 - \frac{v^2}{c^2}} = (8\text{ ly})\sqrt{1 - \frac{(0.8c)^2}{c^2}} = 5\text{ ly}$$

Thus, the travel time measured on her clock is

$$t = \frac{d}{v} = \frac{5\text{ ly}}{0.8c} = 6\text{ yr}$$

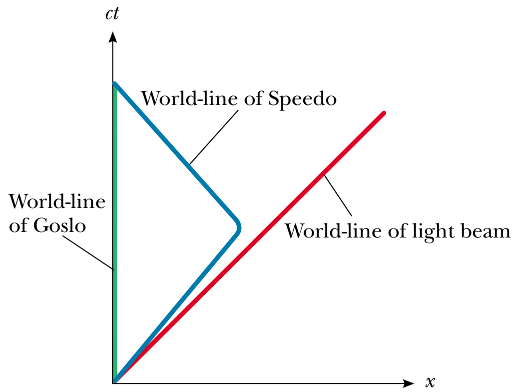


Figure 39.14 The twins paradox on a space–time graph. The twin who stays on the Earth has a world-line along the t axis. The path of the traveling twin through space–time is represented by a world-line that changes direction.

frames, the effect of "offsetting penalties" is seen, and the outcome of the experiment in one frame is the same as the outcome in the other frame!

Space–Time Graphs

It is sometimes helpful to make a *space–time graph,* in which time is the ordinate and displacement is the abscissa. The twins paradox is displayed in such a graph in Figure 39.14. A path through space–time is called a **world-line.** At the origin, the world-lines of Speedo and Goslo coincide because the twins are in the same location at the same time. After Speedo leaves on his trip, his world-line diverges from that of his brother. At their reunion, the two world-lines again come together. Note that Goslo's world-line is vertical, indicating no displacement from his original location. Also note that it would be impossible for Speedo to have a world-line that crossed the path of a light beam that left the Earth when he did. To do so would require him to have a speed greater than c.

World-lines for light beams are diagonal lines on space–time graphs, typically drawn at 45° to the right or left of vertical, depending on whether the light beam is traveling in the direction of increasing or decreasing x. These two world-lines means that all possible future events for Goslo and Speedo lie within two 45° lines extending from the origin. Either twin's presence at an event outside this "light cone" would require that twin to move at a speed greater than c, which, as we shall see in Section 39.5, is not possible. Also, the only past events that Goslo and Speedo could have experienced occurred within two similar 45° world-lines that approach the origin from below the x axis.

Quick Quiz 39.6

How is acceleration indicated on a space–time graph?

The Relativistic Doppler Effect

Another important consequence of time dilation is the shift in frequency found for light emitted by atoms in motion as opposed to light emitted by atoms at rest. This phenomenon, known as the Doppler effect, was introduced in Chapter 17 as it pertains to sound waves. In the case of sound, the motion of the source with respect to the medium of propagation can be distinguished from

the motion of the observer with respect to the medium. Light waves must be analyzed differently, however, because they require no medium of propagation, and no method exists for distinguishing the motion of a light source from the motion of the observer.

If a light source and an observer approach each other with a relative speed v, the frequency f_{obs} measured by the observer is

$$f_{obs} = \frac{\sqrt{1 + v/c}}{\sqrt{1 - v/c}} f_{source} \quad \textbf{(39.10)}$$

where f_{source} is the frequency of the source measured in its rest frame. Note that this relativistic Doppler shift formula, unlike the Doppler shift formula for sound, depends only on the relative speed v of the source and observer and holds for relative speeds as great as c. As you might expect, the formula predicts that $f_{obs} > f_{source}$ when the source and observer approach each other. We obtain the expression for the case in which the source and observer recede from each other by replacing v with $-v$ in Equation 39.10.

The most spectacular and dramatic use of the relativistic Doppler effect is the measurement of shifts in the frequency of light emitted by a moving astronomical object such as a galaxy. Spectral lines normally found in the extreme violet region for galaxies at rest with respect to the Earth are shifted by about 100 nm toward the red end of the spectrum for distant galaxies—indicating that these galaxies are *receding* from us. The American astronomer Edwin Hubble (1889–1953) performed extensive measurements of this *red shift* to confirm that most galaxies are moving away from us, indicating that the Universe is expanding.

39.5 THE LORENTZ TRANSFORMATION EQUATIONS

We have seen that the Galilean transformation equations are not valid when v approaches the speed of light. In this section, we state the correct transformation equations that apply for all speeds in the range $0 \le v < c$.

Suppose that an event that occurs at some point P is reported by two observers, one at rest in a frame S and the other in a frame S′ that is moving to the right with speed v, as in Figure 39.15. The observer in S reports the event with space–time coordinates (x, y, z, t), and the observer in S′ reports the same event using the coordinates (x', y', z', t'). We would like to find a relationship between these coordinates that is valid for all speeds.

The equations that are valid from $v = 0$ to $v = c$ and enable us to transform coordinates from S to S′ are the **Lorentz transformation equations:**

Lorentz transformation equations for $S \rightarrow S'$

$$x' = \gamma(x - vt)$$
$$y' = y$$

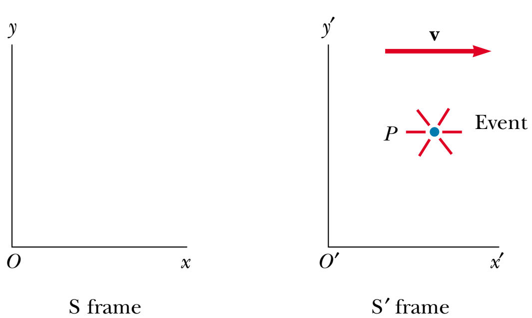


Figure 39.15 An event that occurs at some point P is observed by two persons, one at rest in the S frame and the other in the S′ frame, which is moving to the right with a speed v.

$$z' = z \tag{39.11}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

These transformation equations were developed by Hendrik A. Lorentz (1853–1928) in 1890 in connection with electromagnetism. However, it was Einstein who recognized their physical significance and took the bold step of interpreting them within the framework of the special theory of relativity.

Note the difference between the Galilean and Lorentz time equations. In the Galilean case, $t = t'$, but in the Lorentz case the value for t' assigned to an event by an observer O' standing at the origin of the S′ frame in Figure 39.15 depends both on the time t and on the coordinate x as measured by an observer O standing in the S frame. This is consistent with the notion that an event is characterized by four space–time coordinates (x, y, z, t). In other words, in relativity, space and time are not separate concepts but rather are closely interwoven with each other.

If we wish to transform coordinates in the S′ frame to coordinates in the S frame, we simply replace v by $-v$ and interchange the primed and unprimed coordinates in Equations 39.11:

Inverse Lorentz transformation equations for S′ → S

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned} \tag{39.12}$$

When $v \ll c$, the Lorentz transformation equations should reduce to the Galilean equations. To verify this, note that as v approaches zero, $v/c \ll 1$ and $v^2/c^2 \ll 1$; thus, $\gamma = 1$, and Equations 39.11 reduce to the Galilean space–time transformation equations:

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

In many situations, we would like to know the difference in coordinates between two events or the time interval between two events as seen by observers O and O'. We can accomplish this by writing the Lorentz equations in a form suitable for describing pairs of events. From Equations 39.11 and 39.12, we can express the differences between the four variables x, x', t, and t' in the form

$$\left.\begin{aligned} \Delta x' &= \gamma(\Delta x - v\,\Delta t) \\ \Delta t' &= \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right) \end{aligned}\right\} \mathrm{S} \rightarrow \mathrm{S}' \tag{39.13}$$

$$\left.\begin{aligned} \Delta x &= \gamma(\Delta x' + v\,\Delta t') \\ \Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \end{aligned}\right\} \mathrm{S}' \rightarrow \mathrm{S} \tag{39.14}$$

[5] Although relative motion of the two frames along the x axis does not change the y and z coordinates of an object, it does change the y and z velocity components of an object moving in either frame, as we shall soon see.

EXAMPLE 39.6 Simultaneity and Time Dilation Revisited

Use the Lorentz transformation equations in difference form to show that (a) simultaneity is not an absolute concept and that (b) moving clocks run more slowly than stationary clocks.

Solution (a) Suppose that two events are simultaneous according to a moving observer O', such that $\Delta t' = 0$. From the expression for Δt given in Equation 39.14, we see that in this case the time interval Δt measured by a stationary observer O is $\Delta t = \gamma v \, \Delta x'/c^2$. That is, the time interval for the same two events as measured by O is nonzero, and so the events do not appear to be simultaneous to O.

(b) Suppose that observer O' finds that two events occur at the same place ($\Delta x' = 0$) but at different times ($\Delta t' \neq 0$). In this situation, the expression for Δt given in Equation 39.14 becomes $\Delta t = \gamma \, \Delta t'$. This is the equation for time dilation found earlier (Eq. 39.7), where $\Delta t' = \Delta t_p$ is the proper time measured by a clock located in the moving frame of observer O'.

Exercise Use the Lorentz transformation equations in difference form to confirm that $L = L_p/\gamma$ (Eq. 39.9).

where $\Delta x' = x'_2 - x'_1$ and $\Delta t' = t'_2 - t'_1$ are the differences measured by observer O' and $\Delta x = x_2 - x_1$ and $\Delta t = t_2 - t_1$ are the differences measured by observer O. (We have not included the expressions for relating the y and z coordinates because they are unaffected by motion along the x direction.[5])

Derivation of the Lorentz Velocity Transformation Equation

Once again S is our stationary frame of reference, and S′ is our frame moving at a speed v relative to S. Suppose that an object has a speed u'_x measured in the S′ frame, where

$$u'_x = \frac{dx'}{dt'} \tag{39.15}$$

Using Equation 39.11, we have

$$dx' = \gamma(dx - v\,dt)$$

$$dt' = \gamma\left(dt - \frac{v}{c^2}\,dx\right)$$

Substituting these values into Equation 39.15 gives

$$u'_x = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \dfrac{v}{c^2}\,dx} = \frac{\dfrac{dx}{dt} - v}{1 - \dfrac{v}{c^2}\dfrac{dx}{dt}}$$

But dx/dt is just the velocity component u_x of the object measured by an observer in S, and so this expression becomes

Lorentz velocity transformation equation for S → S′

$$u'_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \tag{39.16}$$

If the object has velocity components along the y and z axes, the components as measured by an observer in S′ are

The speed of light is the speed limit of the Universe. It is the maximum possible speed for energy transfer and for information transfer. Any object with mass must move at a lower speed.

$$u_y' = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \quad \text{and} \quad u_z' = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad \textbf{(39.17)}$$

Lorentz velocity transformation equations for $S' \rightarrow S$

Note that u_y' and u_z' do not contain the parameter v in the numerator because the relative velocity is along the x axis.

When u_x and v are both much smaller than c (the nonrelativistic case), the denominator of Equation 39.16 approaches unity, and so $u_x' \approx u_x - v$, which is the Galilean velocity transformation equation. In the other extreme, when $u_x = c$, Equation 39.16 becomes

$$u_x' = \frac{c - v}{1 - \frac{cv}{c^2}} = \frac{c\left(1 - \frac{v}{c}\right)}{1 - \frac{v}{c}} = c$$

From this result, we see that an object moving with a speed c relative to an observer in S also has a speed c relative to an observer in S′—independent of the relative motion of S and S′. Note that this conclusion is consistent with Einstein's second postulate—that the speed of light must be c relative to all inertial reference frames. Furthermore, the speed of an object can never exceed c. That is, the speed of light is the ultimate speed. We return to this point later when we consider the energy of a particle.

EXAMPLE 39.7 Relative Velocity of Spaceships

Two spaceships A and B are moving in opposite directions, as shown in Figure 39.16. An observer on the Earth measures the speed of ship A to be $0.750c$ and the speed of ship B to be $0.850c$. Find the velocity of ship B as observed by the crew on ship A.

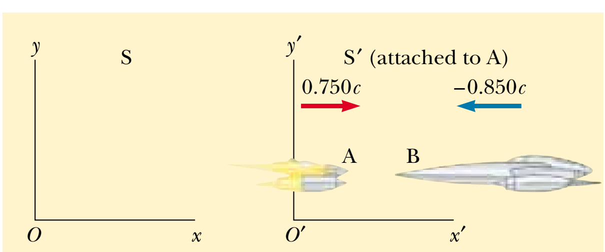


Figure 39.16 Two spaceships A and B move in opposite directions. The speed of B relative to A is *less* than c and is obtained from the relativistic velocity transformation equation.

Solution We can solve this problem by taking the S′ frame as being attached to ship A, so that $v = 0.750c$ relative to the Earth (the S frame). We can consider ship B as moving with a velocity $u_x = -0.850c$ relative to the Earth. Hence, we can obtain the velocity of ship B relative to ship A by using Equation 39.16:

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{-0.850c - 0.750c}{1 - \frac{(-0.850c)(0.750c)}{c^2}} = \boxed{-0.977c}$$

The negative sign indicates that ship B is moving in the negative x direction as observed by the crew on ship A. Note that the speed is less than c. That is, a body whose speed is less than c in one frame of reference must have a speed less than c in any other frame. (If the Galilean velocity transformation equation were used in this example, we would find that $u_x' = u_x - v = -0.850c - 0.750c = -1.60c$, which is impossible. The Galilean transformation equation does not work in relativistic situations.)

EXAMPLE 39.8 The Speeding Motorcycle

Imagine a motorcycle moving with a speed $0.80c$ past a stationary observer, as shown in Figure 39.17. If the rider tosses a ball in the forward direction with a speed of $0.70c$ relative to himself, what is the speed of the ball relative to the stationary observer?

Solution The speed of the motorcycle relative to the stationary observer is $v = 0.80c$. The speed of the ball in the frame of reference of the motorcyclist is $u'_x = 0.70c$. Therefore, the speed u_x of the ball relative to the stationary observer is

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{0.70c + 0.80c}{1 + \frac{(0.70c)(0.80c)}{c^2}} = \boxed{0.96c}$$

Exercise Suppose that the motorcyclist turns on the headlight so that a beam of light moves away from him with a speed c in the forward direction. What does the stationary observer measure for the speed of the light?

Answer c.

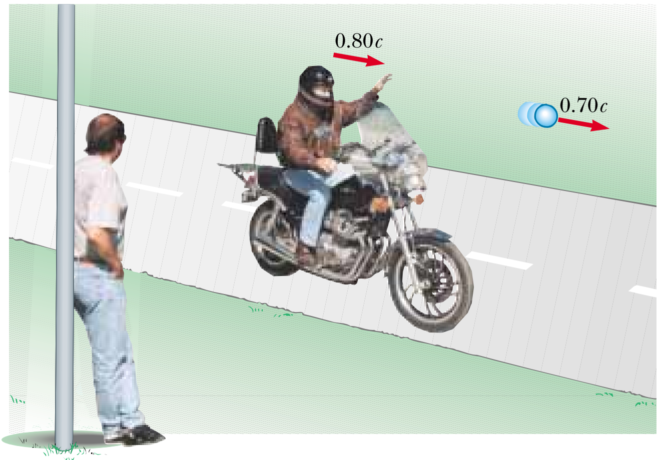


Figure 39.17 A motorcyclist moves past a stationary observer with a speed of $0.80c$ and throws a ball in the direction of motion with a speed of $0.70c$ relative to himself.

EXAMPLE 39.9 Relativistic Leaders of the Pack

Two motorcycle pack leaders named David and Emily are racing at relativistic speeds along perpendicular paths, as shown in Figure 39.18. How fast does Emily recede as seen by David over his right shoulder?

Solution Figure 39.18 represents the situation as seen by a police officer at rest in frame S, who observes the following:

David: $u_x = 0.75c \qquad u_y = 0$

Emily: $u_x = 0 \qquad u_y = -0.90c$

To calculate Emily's speed of recession as seen by David, we take S′ to move along with David and then calculate u'_x and u'_y for Emily using Equations 39.16 and 39.17:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{0 - 0.75c}{1 - \frac{(0)(0.75c)}{c^2}} = -0.75c$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} = \frac{\sqrt{1 - \frac{(0.75c)^2}{c^2}}\,(-0.90c)}{\left(1 - \frac{(0)(0.75c)}{c^2}\right)} = -0.60c$$

Thus, the speed of Emily as observed by David is

$$u' = \sqrt{(u'_x)^2 + (u'_y)^2} = \sqrt{(-0.75c)^2 + (-0.60c)^2} = \boxed{0.96c}$$

Note that this speed is less than c, as required by the special theory of relativity.

Exercise Use the Galilean velocity transformation equation to calculate the classical speed of recession for Emily as observed by David.

Answer $1.2c$.

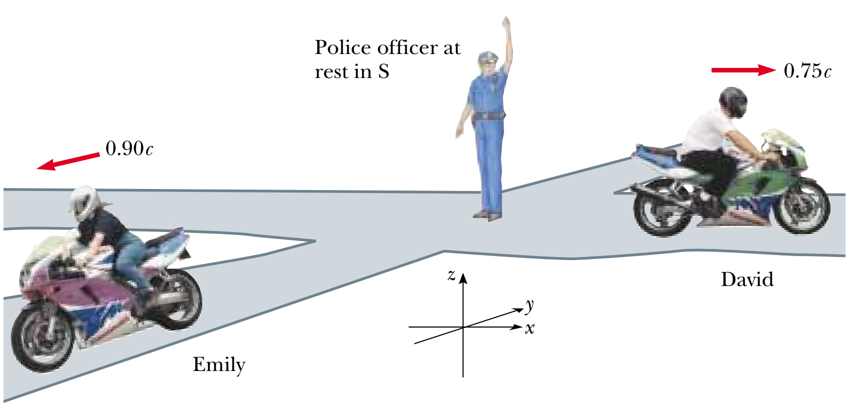


Figure 39.18 David moves to the east with a speed $0.75c$ relative to the police officer, and Emily travels south at a speed $0.90c$ relative to the officer.

To obtain u_x in terms of u'_x, we replace v by $-v$ in Equation 39.16 and interchange the roles of u_x and u'_x:

$$u_x = \frac{u'_x + v}{1 + \dfrac{u'_x v}{c^2}} \tag{39.18}$$

39.6 RELATIVISTIC LINEAR MOMENTUM AND THE RELATIVISTIC FORM OF NEWTON'S LAWS

We have seen that in order to describe properly the motion of particles within the framework of the special theory of relativity, we must replace the Galilean transformation equations by the Lorentz transformation equations. Because the laws of physics must remain unchanged under the Lorentz transformation, we must generalize Newton's laws and the definitions of linear momentum and energy to conform to the Lorentz transformation equations and the principle of relativity. These generalized definitions should reduce to the classical (nonrelativistic) definitions for $v \ll c$.

First, recall that the law of conservation of linear momentum states that when two isolated objects collide, their combined total momentum remains constant. Suppose that the collision is described in a reference frame S in which linear momentum is conserved. If we calculate the velocities in a second reference frame S′ using the Lorentz velocity transformation equation and the classical definition of linear momentum, $\mathbf{p} = m\mathbf{u}$ (where $\mathbf{u}$ is the velocity of either object), we find that linear momentum is *not* conserved in S′. However, because the laws of physics are the same in all inertial frames, linear momentum must be conserved in all frames. In view of this condition and assuming that the Lorentz velocity transformation equation is correct, we must modify the definition of linear momentum to satisfy the following conditions:

Definition of relativistic linear momentum

- Linear momentum $\mathbf{p}$ must be conserved in all collisions.
- The relativistic value calculated for $\mathbf{p}$ must approach the classical value $m\mathbf{u}$ as $\mathbf{u}$ approaches zero.

For any particle, the correct relativistic equation for linear momentum that satisfies these conditions is

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \gamma m\mathbf{u} \tag{39.19}$$

where $\mathbf{u}$ is the velocity of the particle and m is the mass of the particle. When u is much less than c, $\gamma = (1 - u^2/c^2)^{-1/2}$ approaches unity and $\mathbf{p}$ approaches $m\mathbf{u}$. Therefore, the relativistic equation for $\mathbf{p}$ does indeed reduce to the classical expression when u is much smaller than c.

The relativistic force $\mathbf{F}$ acting on a particle whose linear momentum is $\mathbf{p}$ is defined as

$$\mathbf{F} \equiv \frac{d\mathbf{p}}{dt} \tag{39.20}$$

where $\mathbf{p}$ is given by Equation 39.19. This expression, which is the relativistic form of Newton's second law, is reasonable because it preserves classical mechanics in

EXAMPLE 39.10 Linear Momentum of an Electron

An electron, which has a mass of 9.11×10^{-31} kg, moves with a speed of $0.750c$. Find its relativistic momentum and compare this value with the momentum calculated from the classical expression.

Solution Using Equation 39.19 with $u = 0.750c$, we have

$$p = \frac{m_e u}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$= \frac{(9.11 \times 10^{-31}\ \text{kg})(0.750 \times 3.00 \times 10^8\ \text{m/s})}{\sqrt{1 - \frac{(0.750c)^2}{c^2}}}$$

$$= \boxed{3.10 \times 10^{-22}\ \text{kg}\cdot\text{m/s}}$$

The (incorrect) classical expression gives

$$p_{\text{classical}} = m_e u = 2.05 \times 10^{-22}\ \text{kg}\cdot\text{m/s}$$

Hence, the correct relativistic result is 50% greater than the classical result!

the limit of low velocities and requires conservation of linear momentum for an isolated system ($\mathbf{F} = 0$) both relativistically and classically.

It is left as an end-of-chapter problem (Problem 63) to show that under relativistic conditions, the acceleration **a** of a particle decreases under the action of a constant force, in which case $a \propto (1 - u^2/c^2)^{3/2}$. From this formula, note that as the particle's speed approaches c, the acceleration caused by any finite force approaches zero. Hence, it is impossible to accelerate a particle from rest to a speed $u \geq c$.

39.7 RELATIVISTIC ENERGY

We have seen that the definition of linear momentum and the laws of motion require generalization to make them compatible with the principle of relativity. This implies that the definition of kinetic energy must also be modified.

To derive the relativistic form of the work–kinetic energy theorem, let us first use the definition of relativistic force, Equation 39.20, to determine the work done on a particle by a force F:

$$W = \int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} \frac{dp}{dt}\,dx \tag{39.21}$$

for force and motion both directed along the x axis. In order to perform this integration and find the work done on the particle and the relativistic kinetic energy as a function of u, we first evaluate dp/dt:

$$\frac{dp}{dt} = \frac{d}{dt}\frac{mu}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{m(du/dt)}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}$$

Substituting this expression for dp/dt and $dx = u\,dt$ into Equation 39.21 gives

$$W = \int_0^t \frac{m(du/dt)u\,dt}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}} = m\int_0^u \frac{u}{\left(1 - \frac{u^2}{c^2}\right)^{3/2}}\,du$$

where we use the limits 0 and u in the rightmost integral because we have assumed

that the particle is accelerated from rest to some final speed u. Evaluating the integral, we find that

Relativistic kinetic energy

$$W = \frac{mc^2}{\sqrt{1 - \dfrac{u^2}{c^2}}} - mc^2 \quad \textbf{(39.22)}$$

Recall from Chapter 7 that the work done by a force acting on a particle equals the change in kinetic energy of the particle. Because of our assumption that the initial speed of the particle is zero, we know that the initial kinetic energy is zero. We therefore conclude that the work W is equivalent to the relativistic kinetic energy K:

$$K = \frac{mc^2}{\sqrt{1 - \dfrac{u^2}{c^2}}} - mc^2 = \gamma mc^2 - mc^2 \quad \textbf{(39.23)}$$

This equation is routinely confirmed by experiments using high-energy particle accelerators.

At low speeds, where $u/c \ll 1$, Equation 39.23 should reduce to the classical expression $K = \frac{1}{2}mu^2$. We can check this by using the binomial expansion $(1 - x^2)^{-1/2} \approx 1 + \frac{1}{2}x^2 + \ldots$ for $x \ll 1$, where the higher-order powers of x are neglected in the expansion. In our case, $x = u/c$, so that

$$\frac{1}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{u^2}{c^2}$$

Substituting this into Equation 39.23 gives

$$K \approx mc^2\left(1 + \frac{1}{2}\frac{u^2}{c^2}\right) - mc^2 = \frac{1}{2}mu^2$$

Definition of total energy

which is the classical expression for kinetic energy. A graph comparing the relativistic and nonrelativistic expressions is given in Figure 39.19. In the relativistic case, the particle speed never exceeds c, regardless of the kinetic energy. The two curves are in good agreement when $u \ll c$.

The constant term mc^2 in Equation 39.23, which is independent of the speed of the particle, is called the **rest energy** E_R of the particle (see Section 8.9). The term γmc^2, which does depend on the particle speed, is therefore the sum of the kinetic and rest energies. We define γmc^2 to be the **total energy** E:

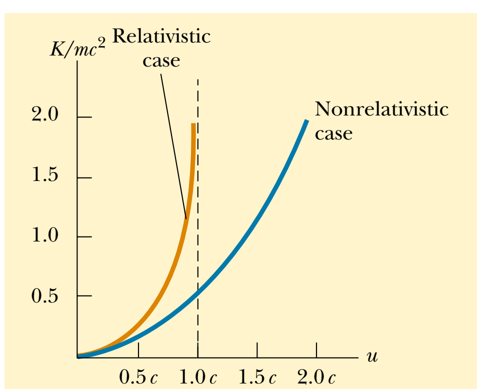


Figure 39.19 A graph comparing relativistic and nonrelativistic kinetic energy. The energies are plotted as a function of speed. In the relativistic case, u is always less than c.

$$\text{Total energy} = \text{kinetic energy} + \text{rest energy}$$

$$E = \gamma mc^2 = K + mc^2 \tag{39.24}$$

or

$$E = \frac{mc^2}{\sqrt{1 - \dfrac{u^2}{c^2}}} \tag{39.25}$$

This is Einstein's famous equation about mass–energy equivalence.

The relationship $E = K + mc^2$ shows that **mass is a form of energy,** where c^2 in the rest energy term is just a constant conversion factor. This expression also shows that a small mass corresponds to an enormous amount of energy, a concept fundamental to nuclear and elementary-particle physics.

Energy–momentum relationship

In many situations, the linear momentum or energy of a particle is measured rather than its speed. It is therefore useful to have an expression relating the total energy E to the relativistic linear momentum p. This is accomplished by using the expressions $E = \gamma mc^2$ and $p = \gamma mu$. By squaring these equations and subtracting, we can eliminate u (Problem 39). The result, after some algebra, is[6]

$$E^2 = p^2c^2 + (mc^2)^2 \tag{39.26}$$

When the particle is at rest, $p = 0$ and so $E = E_R = mc^2$. For particles that have zero mass, such as photons, we set $m = 0$ in Equation 39.26 and see that

$$E = pc \tag{39.27}$$

This equation is an exact expression relating total energy and linear momentum for photons, which always travel at the speed of light.

Finally, note that because the mass m of a particle is independent of its motion, m must have the same value in all reference frames. For this reason, m is often called the **invariant mass.** On the other hand, because the total energy and linear momentum of a particle both depend on velocity, these quantities depend on the reference frame in which they are measured.

Because m is a constant, we conclude from Equation 39.26 that the quantity $E^2 - p^2c^2$ must have the same value in all reference frames. That is, $E^2 - p^2c^2$ is invariant under a Lorentz transformation. (Equations 39.26 and 39.27 do not make provision for potential energy.)

When we are dealing with subatomic particles, it is convenient to express their

Example 39.11 The Energy of a Speedy Electron

An electron in a television picture tube typically moves with a speed $u = 0.250c$. Find its total energy and kinetic energy in electron volts.

Solution Using the fact that the rest energy of the electron is 0.511 MeV together with Equation 39.25, we have

$$E = \frac{m_ec^2}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \frac{0.511\text{ MeV}}{\sqrt{1 - \dfrac{(0.250c)^2}{c^2}}}$$

$$= 1.03(0.511\text{ MeV}) = \boxed{0.528\text{ MeV}}$$

This is 3% greater than the rest energy.

We obtain the kinetic energy by subtracting the rest energy from the total energy:

$$K = E - m_ec^2 = 0.528\text{ MeV} - 0.511\text{ MeV} = \boxed{0.017\text{ MeV}}$$

[6] One way to remember this relationship is to draw a right triangle having a hypotenuse of length E and legs of lengths pc and mc^2.

Example 39.12 The Energy of a Speedy Proton

(a) Find the rest energy of a proton in electron volts.

Solution

$$E_R = m_pc^2 = (1.67 \times 10^{-27}\ \text{kg})(3.00 \times 10^8\ \text{m/s})^2$$
$$= (1.50 \times 10^{-10}\ \text{J})(1.00\ \text{eV}/1.60 \times 10^{-19}\ \text{J})$$
$$= \boxed{938\ \text{MeV}}$$

(b) If the total energy of a proton is three times its rest energy, with what speed is the proton moving?

Solution Equation 39.25 gives

$$E = 3m_pc^2 = \frac{m_pc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$3 = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Solving for u gives

$$1 - \frac{u^2}{c^2} = \frac{1}{9}$$

$$\frac{u^2}{c^2} = \frac{8}{9}$$

$$u = \frac{\sqrt{8}}{3}c = \boxed{2.83 \times 10^8\ \text{m/s}}$$

(c) Determine the kinetic energy of the proton in electron volts.

Solution From Equation 39.24,

$$K = E - m_pc^2 = 3m_pc^2 - m_pc^2 = 2m_pc^2$$

Because $m_pc^2 = 938$ MeV, $K = \boxed{1\ 880\ \text{MeV}}$

(d) What is the proton's momentum?

Solution We can use Equation 39.26 to calculate the momentum with $E = 3m_pc^2$:

$$E^2 = p^2c^2 + (m_pc^2)^2 = (3m_pc^2)^2$$
$$p^2c^2 = 9(m_pc^2)^2 - (m_pc^2)^2 = 8(m_pc^2)^2$$
$$p = \sqrt{8}\frac{m_pc^2}{c} = \sqrt{8}\frac{(938\ \text{MeV})}{c} = \boxed{2\ 650\ \text{MeV}/c}$$

The unit of momentum is written MeV/c for convenience.

energy in electron volts because the particles are usually given this energy by acceleration through a potential difference. The conversion factor, as you recall from Equation 25.5, is

$$1\ \text{eV} = 1.602 \times 10^{-19}\ \text{J}$$

For example, the mass of an electron is 9.109×10^{-31} kg. Hence, the rest energy of the electron is

$$m_ec^2 = (9.109 \times 10^{-31}\ \text{kg})(2.9979 \times 10^8\ \text{m/s})^2 = 8.187 \times 10^{-14}\ \text{J}$$
$$= (8.187 \times 10^{-14}\ \text{J})(1\ \text{eV}/1.602 \times 10^{-19}\ \text{J}) = 0.5110\ \text{MeV}$$

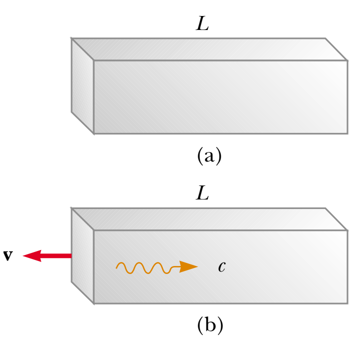


Figure 39.20 (a) A box of length L at rest. (b) When a light pulse directed to the right is emitted at the left end of the box, the box recoils to the left until the pulse strikes the right end.

39.8 Equivalence of Mass and Energy

To understand the equivalence of mass and energy, consider the following thought experiment proposed by Einstein in developing his famous equation $E = mc^2$. Imagine an isolated box of mass M_{box} and length L initially at rest, as shown in Figure 39.20a. Suppose that a pulse of light is emitted from the left side of the box, as depicted in Figure 39.20b. From Equation 39.27, we know that light of energy E carries linear momentum $p = E/c$. Hence, if momentum is to be conserved, the box must recoil to the left with a speed v. If it is assumed that the box is very mas-

sive, the recoil speed is much less than the speed of light, and conservation of momentum gives $M_{\text{box}}v = E/c$, or

$$v = \frac{E}{M_{\text{box}}c}$$

The time it takes the light pulse to move the length of the box is approximately $\Delta t = L/c$. In this time interval, the box moves a small distance Δx to the left, where

$$\Delta x = v\,\Delta t = \frac{EL}{M_{\text{box}}c^2}$$

The light then strikes the right end of the box and transfers its momentum to the box, causing the box to stop. With the box in its new position, its center of mass appears to have moved to the left. However, its center of mass cannot have moved because the box is an isolated system. Einstein resolved this perplexing situation by assuming that in addition to energy and momentum, light also carries mass. If M_{pulse} is the effective mass carried by the pulse of light and if the center of mass of the system (box plus pulse of light) is to remain fixed, then

$$M_{\text{pulse}}L = M_{\text{box}}\Delta x$$

Solving for M_{pulse}, and using the previous expression for Δx, we obtain

$$M_{\text{pulse}} = \frac{M_{\text{box}}\Delta x}{L} = \frac{M_{\text{box}}}{L}\,\frac{EL}{M_{\text{box}}c^2} = \frac{E}{c^2}$$

or

$$E = M_{\text{pulse}}c^2$$

the energy of a system of particles before interaction must equal the energy of the system after interaction, where energy of the ith particle is given by the expression

$$E_i = \frac{m_i c^2}{\sqrt{1 - \dfrac{u_i^2}{c^2}}} = \gamma m_i c^2$$

Conversion of mass–energy

Thus, Einstein reached the profound conclusion that "if a body gives off the energy E in the form of radiation, its mass diminishes by E/c^2, . . ."

Although we derived the relationship $E = mc^2$ for light energy, the equivalence of mass and energy is universal. Equation 39.24, $E = \gamma mc^2$, which represents the total energy of any particle, suggests that even when a particle is at rest ($\gamma = 1$) it still possesses enormous energy because it has mass. Probably the clearest experimental proof of the equivalence of mass and energy occurs in nuclear and elementary particle interactions, where large amounts of energy are released and the energy release is accompanied by a decrease in mass. Because energy and mass are related, we see that the laws of conservation of energy and conservation of mass are one and the same. Simply put, this law states that

The release of enormous quantities of energy from subatomic particles, accompanied by changes in their masses, is the basis of all nuclear reactions. In a conventional nuclear reactor, a uranium nucleus undergoes *fission,* a reaction that creates several lighter fragments having considerable kinetic energy. The com-

bined mass of all the fragments is less than the mass of the parent uranium nucleus by an amount Δm. The corresponding energy Δmc^2 associated with this mass difference is exactly equal to the total kinetic energy of the fragments. This kinetic energy raises the temperature of water in the reactor, converting it to steam for the generation of electric power.

CONCEPTUAL EXAMPLE 39.13

Because mass is a measure of energy, can we conclude that the mass of a compressed spring is greater than the mass of the same spring when it is not compressed?

Solution Recall that when a spring of force constant k is compressed (or stretched) from its equilibrium position a distance x, it stores elastic potential energy $U = kx^2/2$. According to the special theory of relativity, any change in the total energy of a system is equivalent to a change in the mass of the system. Therefore, the mass of a compressed (or stretched) spring is greater than the mass of the spring in its equilibrium position by an amount U/c^2.

EXAMPLE 39.14 Binding Energy of the Deuteron

A deuteron, which is the nucleus of a deuterium atom, contains one proton and one neutron and has a mass of 2.013 553 u. This total deuteron mass is not equal to the sum of the masses of the proton and neutron. Calculate the mass difference and determine its energy equivalence, which is called the *binding energy* of the nucleus.

Solution Using atomic mass units (u), we have

$$m_p = \text{mass of proton} = 1.007\,276 \text{ u}$$

$$m_n = \text{mass of neutron} = 1.008\,665 \text{ u}$$

$$m_p + m_n = 2.015\,941 \text{ u}$$

The mass difference Δm is therefore 0.002 388 u. By definition, 1 u = 1.66×10^{-27} kg, and therefore

$$\Delta m = 0.002\,388 \text{ u} = \boxed{3.96 \times 10^{-30} \text{ kg}}$$

Using $E = \Delta mc^2$, we find that the binding energy is

$$E = \Delta mc^2 = (3.96 \times 10^{-30} \text{ kg})(3.00 \times 10^8 \text{ m/s})^2$$

$$= 3.56 \times 10^{-13} \text{ J} = \boxed{2.23 \text{ MeV}}$$

Therefore, the minimum energy required to separate the proton from the neutron of the deuterium nucleus (the binding energy) is 2.23 MeV.

In the nuclear reaction called *fusion,* two atomic nuclei combine to form a single nucleus. The fusion reaction in which two deuterium nuclei fuse to form a helium nucleus is of major importance in current research and the development of controlled-fusion reactors. The decrease in mass that results from the creation of one helium nucleus from two deuterium nuclei is $\Delta m = 4.25 \times 10^{-29}$ kg. Hence, the corresponding excess energy that results from one fusion reaction is $\Delta mc^2 = 3.83 \times 10^{-12}$ J = 23.9 MeV. To appreciate the magnitude of this result, note that if 1 g of deuterium is converted to helium, the energy released is about 10^{12} J! At the current cost of electrical energy, this quantity of energy would be worth about $70 000.

39.9 RELATIVITY AND ELECTROMAGNETISM

Consider two frames of reference S and S′ that are in relative motion, and assume that a single charge q is at rest in the S′ frame of reference. According to an ob-

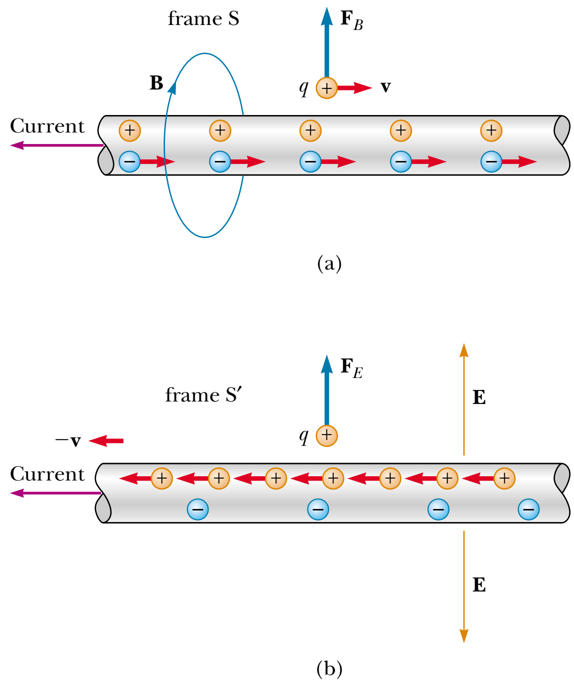


Figure 39.21 (a) In frame S, the positive charge q moves to the right with a velocity **v**, and the current-carrying wire is stationary. A magnetic field **B** surrounds the wire, and charge experiences a *magnetic* force directed away from the wire. (b) In frame S′, the wire moves to the left with a velocity −**v**, and the charge q is stationary. The wire creates an electric field **E**, and the charge experiences an *electric* force directed away from the wire.

server in this frame, an electric field surrounds the charge. However, an observer in frame S says that the charge is in motion and therefore measures both an electric field and a magnetic field. The magnetic field measured by the observer in frame S is created by the moving charge, which constitutes an electric current. In other words, electric and magnetic fields are viewed differently in frames of reference that are moving relative to each other. We now describe one situation that shows how an electric field in one frame of reference is viewed as a magnetic field in another frame of reference.

A positive test charge q is moving parallel to a current-carrying wire with velocity **v** relative to the wire in frame S, as shown in Figure 39.21a. We assume that the net charge on the wire is zero and that the electrons in the wire also move with velocity **v** in a straight line. The leftward current in the wire produces a magnetic field that forms circles around the wire and is directed into the page at the location of the moving test charge. Therefore, a magnetic force $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$ directed away from the wire is exerted on the test charge. However, no electric force acts on the test charge because the net charge on the wire is zero when viewed in this frame.

Now consider the same situation as viewed from frame S′, where the test charge is at rest (Figure 39.21b). In this frame, the positive charges in the wire move to the left, the electrons in the wire are at rest, and the wire still carries a cur-

rent. Because the test charge is not moving in this frame, $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B} = 0$; there is no magnetic force exerted on the test charge when viewed in this frame. However, if a force is exerted on the test charge in frame S′, the frame of the wire, as described earlier, a force must be exerted on it in any other frame. What is the origin of this force in frame S, the frame of the test charge?

The answer to this question is provided by the special theory of relativity. When the situation is viewed in frame S, as in Figure 39.21a, the positive charges are at rest and the electrons in the wire move to the right with a velocity **v**. Because of length contraction, the electrons appear to be closer together than their proper separation. Because there is no net charge on the wire this contracted separation must equal the separation between the stationary positive charges. The situation is quite different when viewed in frame S′, shown in Figure 39.21b. In this frame, the positive charges appear closer together because of length contraction, and the electrons in the wire are at rest with a separation that is greater than that viewed in frame S. Therefore, there is a net positive charge on the wire when viewed in frame S′. This net positive charge produces an electric field pointing away from the wire toward the test charge, and so the test charge experiences an electric force directed away from the wire. Thus, what was viewed as a magnetic field (and a corresponding magnetic force) in the frame of the wire transforms into an electric field (and a corresponding electric force) in the frame of the test charge.

Optional Section

39.10 THE GENERAL THEORY OF RELATIVITY

Up to this point, we have sidestepped a curious puzzle. Mass has two seemingly different properties: a *gravitational attraction* for other masses and an *inertial* property that resists acceleration. To designate these two attributes, we use the subscripts g and i and write

$$\text{Gravitational property} \qquad F_g = m_g g$$

$$\text{Inertial property} \qquad \Sigma F = m_i a$$

The value for the gravitational constant G was chosen to make the magnitudes of m_g and m_i numerically equal. Regardless of how G is chosen, however, the strict proportionality of m_g and m_i has been established experimentally to an extremely high degree: a few parts in 10^{12}. Thus, it appears that gravitational mass and inertial mass may indeed be exactly proportional.

But why? They seem to involve two entirely different concepts: a force of mutual gravitational attraction between two masses, and the resistance of a single mass to being accelerated. This question, which puzzled Newton and many other physicists over the years, was answered when Einstein published his theory of gravitation, known as his *general theory of relativity,* in 1916. Because it is a mathematically complex theory, we offer merely a hint of its elegance and insight.

In Einstein's view, the remarkable coincidence that m_g and m_i seemed to be proportional to each other was evidence of an intimate and basic connection between the two concepts. He pointed out that no mechanical experiment (such as dropping a mass) could distinguish between the two situations illustrated in Figure 39.22a and b. In each case, the dropped briefcase undergoes a downward acceleration g relative to the floor.

Einstein carried this idea further and proposed that *no* experiment, mechanical or otherwise, could distinguish between the two cases. This extension to in-

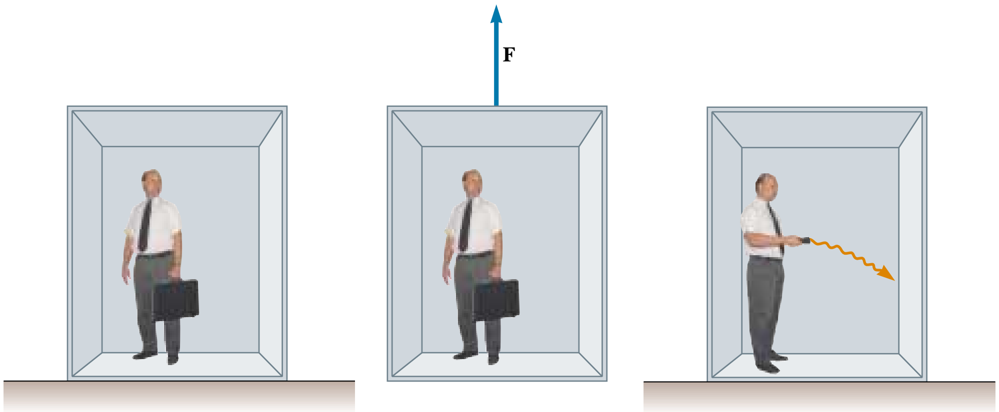


Figure 39.22 (a) The observer is at rest in a uniform gravitational field **g**. (b) The observer is in a region where gravity is negligible, but the frame of reference is accelerated by an external force **F** that produces an acceleration **g**. According to Einstein, the frames of reference in parts (a) and (b) are equivalent in every way. No local experiment can distinguish any difference between the two frames. (c) If parts (a) and (b) are truly equivalent, as Einstein proposed, then a ray of light should bend in a gravitational field.

clude all phenomena (not just mechanical ones) has interesting consequences. For example, suppose that a light pulse is sent horizontally across the elevator. During the time it takes the light to make the trip, the right wall of the elevator has accelerated upward. This causes the light to arrive at a location lower on the wall than the spot it would have hit if the elevator were not accelerating. Thus, in the frame of the elevator, the trajectory of the light pulse bends downward as the elevator accelerates upward to meet it. Because the accelerating elevator cannot be distinguished from a nonaccelerating one located in a gravitational field, Einstein proposed that a beam of light *should also be bent downward by a gravitational field,* as shown in Figure 39.22c. Experiments have verified the effect, although the bending is small. A laser aimed at the horizon falls less than 1 cm after traveling 6 000 km. (No such bending is predicted in Newton's theory of gravitation.)

The two postulates of Einstein's **general theory of relativity** are

- All the laws of nature have the same form for observers in any frame of reference, whether accelerated or not.

This Global Positioning System (GPS) unit incorporates relativistically corrected time calculations in its analysis of signals it receives from orbiting satellites. These corrections allow the unit to determine its position on the Earth's surface to within a few meters. If the corrections were not made, the location error would be about 1 km. *(Courtesy of Trimble Navigation Limited)*

- In the vicinity of any point, a gravitational field is equivalent to an accelerated frame of reference in the absence of gravitational effects. (This is the *principle of equivalence.*)

The second postulate implies that gravitational mass and inertial mass are completely equivalent, not just proportional. What were thought to be two different types of mass are actually identical.

One interesting effect predicted by the general theory is that time scales are altered by gravity. A clock in the presence of gravity runs more slowly than one located where gravity is negligible. Consequently, the frequencies of radiation emitted by atoms in the presence of a strong gravitational field are *red-shifted* to lower frequencies when compared with the same emissions in the presence of a weak field. This gravitational red shift has been detected in spectral lines emitted by atoms in massive stars. It has also been verified on the Earth by comparison of the frequencies of gamma rays (a high-energy form of electromagnetic radiation) emitted from nuclei separated vertically by about 20 m.

Quick Quiz 39.7

Two identical clocks are in the same house, one upstairs in a bedroom and the other downstairs in the kitchen. Which clock runs more slowly?

The second postulate suggests that a gravitational field may be "transformed away" at any point if we choose an appropriate accelerated frame of reference—a freely falling one. Einstein developed an ingenious method of describing the acceleration necessary to make the gravitational field "disappear." He specified a concept, the *curvature of space–time,* that describes the gravitational effect at every point. In fact, the curvature of space–time completely replaces Newton's gravitational theory. According to Einstein, there is no such thing as a gravitational force. Rather, the presence of a mass causes a curvature of space–time in the vicinity of the mass, and this curvature dictates the space–time path that all freely moving objects must follow. In 1979, John Wheeler summarized Einstein's general theory of relativity in a single sentence: "Space tells matter how to move and matter tells space how to curve."

Consider two travelers on the surface of the Earth walking directly toward the

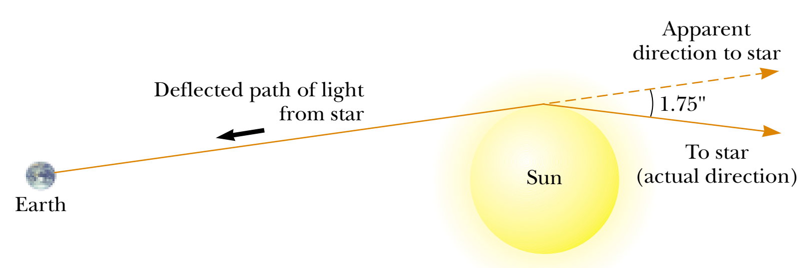
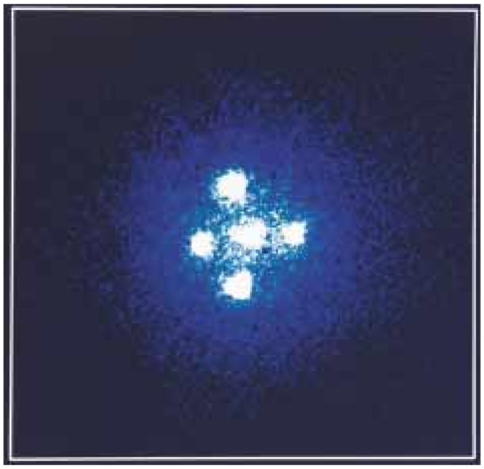


Figure 39.23 Deflection of starlight passing near the Sun. Because of this effect, the Sun or some other remote object can act as a *gravitational lens.* In his general theory of relativity, Einstein calculated that starlight just grazing the Sun's surface should be deflected by an angle of 1.75″.



Einstein's cross. The four bright spots are images of the same galaxy that have been bent around a massive object located between the galaxy and the Earth. The massive object acts like a lens, causing the rays of light that were diverging from the distant galaxy to converge on the Earth. (If the intervening massive object had a uniform mass distribution, we would see a bright ring instead of four spots.)

North Pole but from different starting locations. Even though both say they are walking due north, and thus should be on parallel paths, they see themselves getting closer and closer together, as if they were somehow attracted to each other. The curvature of the Earth causes this effect. In a similar way, what we are used to thinking of as the gravitational attraction between two masses is, in Einstein's view, two masses curving space–time and as a result moving toward each other, much like two bowling balls on a mattress rolling together.

One prediction of the general theory of relativity is that a light ray passing near the Sun should be deflected into the curved space–time created by the Sun's mass. This prediction was confirmed when astronomers detected the bending of starlight near the Sun during a total solar eclipse that occurred shortly after World War I (Fig. 39.23). When this discovery was announced, Einstein became an international celebrity.

If the concentration of mass becomes very great, as is believed to occur when a large star exhausts its nuclear fuel and collapses to a very small volume, a **black hole** may form. Here, the curvature of space–time is so extreme that, within a certain distance from the center of the black hole, all matter and light become trapped.

SUMMARY

The two basic postulates of the special theory of relativity are

- The laws of physics must be the same in all inertial reference frames.
- The speed of light in vacuum has the same value, $c = 3.00 \times 10^8$ m/s, in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

Three consequences of the special theory of relativity are

- Events that are simultaneous for one observer are not simultaneous for another observer who is in motion relative to the first.
- Clocks in motion relative to an observer appear to be slowed down by a factor $\gamma = (1 - v^2/c^2)^{-1/2}$. This phenomenon is known as **time dilation.**
- The length of objects in motion appears to be contracted in the direction of

motion by a factor $1/\gamma = (1 - v^2/c^2)^{1/2}$. This phenomenon is known as **length contraction.**

To satisfy the postulates of special relativity, the Galilean transformation equations must be replaced by the **Lorentz transformation equations:**

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned} \qquad \textbf{(39.11)}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

The relativistic form of the **velocity transformation equation** is

$$u'_x = \frac{u_x - v}{1 - \dfrac{u_x v}{c^2}} \qquad \textbf{(39.16)}$$

where u_x is the speed of an object as measured in the S frame and u'_x is its speed measured in the S′ frame.

The relativistic expression for the **linear momentum** of a particle moving with a velocity $\mathbf{u}$ is

$$\mathbf{p} \equiv \frac{m\mathbf{u}}{\sqrt{1 - \dfrac{u^2}{c^2}}} = \gamma m\mathbf{u} \qquad \textbf{(39.19)}$$

The relativistic expression for the **kinetic energy** of a particle is

QUESTIONS

1. What two speed measurements do two observers in relative motion always agree on?
2. A spaceship in the shape of a sphere moves past an observer on the Earth with a speed $0.5c$. What shape does the observer see as the spaceship moves past?
3. An astronaut moves away from the Earth at a speed close to the speed of light. If an observer on Earth measures the astronaut's dimensions and pulse rate, what changes (if any) would the observer measure? Would the astronaut measure any changes about himself?
4. Two identical clocks are synchronized. One is then put in orbit directed eastward around the Earth while the other remains on Earth. Which clock runs slower? When the moving clock returns to Earth, are the two still synchronized?
5. Two lasers situated on a moving spacecraft are triggered simultaneously. An observer on the spacecraft claims to see the pulses of light simultaneously. What condition is necessary so that a second observer agrees?
6. When we say that a moving clock runs more slowly than a stationary one, does this imply that there is something physically unusual about the moving clock?
7. List some ways our day-to-day lives would change if the speed of light were only 50 m/s.
8. Give a physical argument that shows that it is impossible to accelerate an object of mass m to the speed of light, even if it has a continuous force acting on it.
9. It is said that Einstein, in his teenage years, asked the question, "What would I see in a mirror if I carried it in my hands and ran at the speed of light?" How would you answer this question?
10. Some distant star-like objects, called *quasars,* are receding from us at half the speed of light (or greater). What is the speed of the light we receive from these quasars?
11. How is it possible that photons of light, which have zero mass, have momentum?
12. With regard to reference frames, how does general relativity differ from special relativity?
13. Describe how the results of Example 39.7 would change if, instead of fast spaceships, two ordinary cars were approaching each other at highway speeds.
14. Two objects are identical except that one is hotter than the other. Compare how they respond to identical forces.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 39.1 The Principle of Galilean Relativity

1. A 2 000-kg car moving at 20.0 m/s collides and locks together with a 1 500-kg car at rest at a stop sign. Show that momentum is conserved in a reference frame moving at 10.0 m/s in the direction of the moving car.
2. A ball is thrown at 20.0 m/s inside a boxcar moving along the tracks at 40.0 m/s. What is the speed of the ball relative to the ground if the ball is thrown (a) forward? (b) backward? (c) out the side door?
3. In a laboratory frame of reference, an observer notes that Newton's second law is valid. Show that it is also valid for an observer moving at a constant speed, small compared with the speed of light, relative to the laboratory frame.
4. Show that Newton's second law is *not* valid in a reference frame moving past the laboratory frame of Problem 3 with a constant acceleration.

Section 39.2 The Michelson–Morley Experiment
Section 39.3 Einstein's Principle of Relativity
Section 39.4 Consequences of the Special Theory of Relativity

5. How fast must a meter stick be moving if its length is observed to shrink to 0.500 m?
6. At what speed does a clock have to move if it is to be seen to run at a rate that is one-half the rate of a clock at rest?
7. An astronaut is traveling in a space vehicle that has a speed of $0.500c$ relative to the Earth. The astronaut measures his pulse rate at 75.0 beats per minute. Signals generated by the astronaut's pulse are radioed to Earth when the vehicle is moving in a direction perpendicular to a line that connects the vehicle with an observer on the Earth. What pulse rate does the Earth observer measure? What would be the pulse rate if the speed of the space vehicle were increased to $0.990c$?
8. The proper length of one spaceship is three times that of another. The two spaceships are traveling in the same direction and, while both are passing overhead, an Earth observer measures the two spaceships to have the same length. If the slower spaceship is moving with a speed of $0.350c$, determine the speed of the faster spaceship.
9. An atomic clock moves at 1 000 km/h for 1 h as measured by an identical clock on Earth. How many nanoseconds slow will the moving clock be at the end of the 1-h interval?
10. If astronauts could travel at $v = 0.950c$, we on Earth would say it takes $(4.20/0.950) = 4.42$ yr to reach Alpha Centauri, 4.20 ly away. The astronauts disagree. (a) How much time passes on the astronauts' clocks? (b) What distance to Alpha Centauri do the astronauts measure?
11. WEB A spaceship with a proper length of 300 m takes 0.750 μs to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.
12. A spaceship of proper length L_p takes time t to pass an Earth observer. Determine the speed of this spaceship as measured by the Earth observer.
13. A muon formed high in the Earth's atmosphere travels at speed $v = 0.990c$ for a distance of 4.60 km before it decays into an electron, a neutrino, and an antineutrino $(\mu^- \rightarrow e^- + \nu + \bar{\nu})$. (a) How long does the muon live, as measured in its reference frame? (b) How far does the muon travel, as measured in its frame?
14. **Review Problem.** In 1962, when Mercury astronaut Scott Carpenter orbited the Earth 22 times, the press stated that for each orbit he aged 2 millionths of a second less than he would have had he remained on Earth. (a) Assuming that he was 160 km above the Earth in a circular orbit, determine the time difference between someone on Earth and the orbiting astronaut for the 22 orbits. You will need to use the approximation $\sqrt{1 - x} \approx 1 - x/2$ for small x. (b) Did the press report accurate information? Explain.
15. The pion has an average lifetime of 26.0 ns when at rest. In order for it to travel 10.0 m, how fast must it move?
16. For what value of v does $\gamma = 1.01$? Observe that for speeds less than this value, time dilation and length contraction are less-than-one-percent effects.
17. A friend passes by you in a spaceship traveling at a high speed. He tells you that his ship is 20.0 m long and that the identically constructed ship you are sitting in is 19.0 m long. According to your observations, (a) how long is your ship, (b) how long is your friend's ship, and (c) what is the speed of your friend's ship?
18. An interstellar space probe is launched from Earth. After a brief period of acceleration it moves with a constant velocity, 70.0% of the speed of light. Its nuclear-powered batteries supply the energy to keep its data transmitter active continuously. The batteries have a lifetime of 15.0 yr as measured in a rest frame. (a) How long do the batteries on the space probe last as measured by Mission Control on Earth? (b) How far is the probe from Earth when its batteries fail, as measured by Mission Control? (c) How far is the probe from Earth when its batteries fail, as measured by its built-in trip odometer? (d) For what total time after launch are data

received from the probe by Mission Control? Note that radio waves travel at the speed of light and fill the space between the probe and Earth at the time of battery failure.

19. **Review Problem.** An alien civilization occupies a brown dwarf, nearly stationary relative to the Sun, several lightyears away. The extraterrestrials have come to love original broadcasts of *The Ed Sullivan Show,* on our television channel 2, at carrier frequency 57.0 MHz. Their line of sight to us is in the plane of the Earth's orbit. Find the difference between the highest and lowest frequencies they receive due to the Earth's orbital motion around the Sun.

20. Police radar detects the speed of a car (Fig. P39.20) as follows: Microwaves of a precisely known frequency are broadcast toward the car. The moving car reflects the microwaves with a Doppler shift. The reflected waves are received and combined with an attenuated version of the transmitted wave. Beats occur between the two microwave signals. The beat frequency is measured. (a) For an electromagnetic wave reflected back to its source from a mirror approaching at speed v, show that the reflected wave has frequency

$$f = f_{\text{source}} \frac{c+v}{c-v}$$

where f_{source} is the source frequency. (b) When v is much less than c, the beat frequency is much less than the transmitted frequency. In this case, use the approximation $f + f_{\text{source}} \cong 2f_{\text{source}}$ and show that the beat frequency can be written as $f_b = 2v/\lambda$. (c) What beat frequency is measured for a car speed of 30.0 m/s if the microwaves have frequency 10.0 GHz? (d) If the beat frequency measurement is accurate to ± 5 Hz, how accurate is the velocity measurement?

21. *The red shift.* A light source recedes from an observer with a speed v_{source}, which is small compared with c. (a) Show that the fractional shift in the measured wavelength is given by the approximate expression

$$\frac{\Delta\lambda}{\lambda} \approx \frac{v_{\text{source}}}{c}$$

This phenomenon is known as the red shift because the visible light is shifted toward the red. (b) Spectroscopic measurements of light at $\lambda = 397$ nm coming from a galaxy in Ursa Major reveal a red shift of 20.0 nm. What is the recessional speed of the galaxy?

Figure P39.20 *(Trent Steffler/David R. Frazier Photolibrary)*

Section 39.5 The Lorentz Transformation Equations

22. A spaceship travels at $0.750c$ relative to Earth. If the spaceship fires a small rocket in the forward direction, how fast (relative to the ship) must it be fired for it to travel at $0.950c$ relative to Earth?

WEB 23. Two jets of material from the center of a radio galaxy fly away in opposite directions. Both jets move at $0.750c$ relative to the galaxy. Determine the speed of one jet relative to that of the other.

24. A moving rod is observed to have a length of 2.00 m, and to be oriented at an angle of 30.0° with respect to the direction of motion (Fig. P39.24). The rod has a speed of $0.995c$. (a) What is the proper length of the rod? (b) What is the orientation angle in the proper frame?

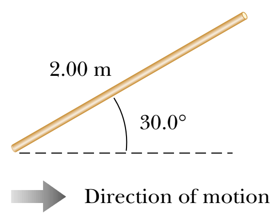


Figure P39.24

25. A Klingon space ship moves away from the Earth at a speed of $0.800c$ (Fig. P39.25). The starship *Enterprise* pursues at a speed of $0.900c$ relative to the Earth. Observers on Earth see the *Enterprise* overtaking the Klingon ship at a relative speed of $0.100c$. With what speed is the *Enterprise* overtaking the Klingon ship as seen by the crew of the *Enterprise*?

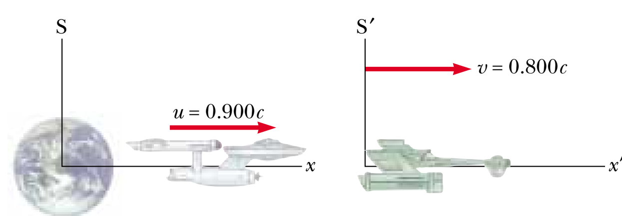


Figure P39.25

26. A red light flashes at position $x_R = 3.00$ m and time $t_R = 1.00 \times 10^{-9}$ s, and a blue light flashes at $x_B = 5.00$ m and $t_B = 9.00 \times 10^{-9}$ s (all values are measured in the S reference frame). Reference frame S′ has its origin at the same point as S at $t = t' = 0$; frame S′ moves constantly to the right. Both flashes are observed to occur at the same place in S′. (a) Find the relative velocity between S and S′. (b) Find the location of the two flashes in frame S′. (c) At what time does the red flash occur in the S′ frame?

Section 39.6 Relativistic Linear Momentum and the Relativistic Form of Newton's Laws

27. Calculate the momentum of an electron moving with a speed of (a) $0.010\,0c$, (b) $0.500c$, (c) $0.900c$.
28. The nonrelativistic expression for the momentum of a particle, $p = mu$, can be used if $u \ll c$. For what speed does the use of this formula yield an error in the momentum of (a) 1.00 percent and (b) 10.0 percent?
29. A golf ball travels with a speed of 90.0 m/s. By what fraction does its relativistic momentum p differ from its classical value mu? That is, find the ratio $(p - mu)/mu$.
30. Show that the speed of an object having momentum p and mass m is

$$u = \frac{c}{\sqrt{1 + (mc/p)^2}}$$

WEB 31. An unstable particle at rest breaks into two fragments of *unequal* mass. The mass of the lighter fragment is 2.50×10^{-28} kg, and that of the heavier fragment is 1.67×10^{-27} kg. If the lighter fragment has a speed of $0.893c$ after the breakup, what is the speed of the heavier fragment?

Section 39.7 Relativistic Energy

32. Determine the energy required to accelerate an electron (a) from $0.500c$ to $0.900c$ and (b) from $0.900c$ to $0.990c$.
33. Find the momentum of a proton in MeV/c units if its total energy is twice its rest energy.
34. Show that, for any object moving at less than one-tenth the speed of light, the relativistic kinetic energy agrees with the result of the classical equation $K = mu^2/2$ to within less than 1%. Thus, for most purposes, the classical equation is good enough to describe these objects, whose motion we call *nonrelativistic.*

WEB 35. A proton moves at $0.950c$. Calculate its (a) rest energy, (b) total energy, and (c) kinetic energy.
36. An electron has a kinetic energy five times greater than its rest energy. Find (a) its total energy and (b) its speed.
37. A cube of steel has a volume of 1.00 cm^3 and a mass of 8.00 g when at rest on the Earth. If this cube is now given a speed $u = 0.900c$, what is its density as measured by a stationary observer? Note that relativistic density is E_R/c^2V.
38. An unstable particle with a mass of 3.34×10^{-27} kg is initially at rest. The particle decays into two fragments that fly off with velocities of $0.987c$ and $-0.868c$. Find the masses of the fragments. (*Hint:* Conserve both mass–energy and momentum.)
39. Show that the energy–momentum relationship $E^2 = p^2c^2 + (mc^2)^2$ follows from the expressions $E = \gamma mc^2$ and $p = \gamma mu$.
40. A proton in a high-energy accelerator is given a kinetic energy of 50.0 GeV. Determine (a) its momentum and (b) its speed.
41. In a typical color television picture tube, the electrons are accelerated through a potential difference of 25 000 V. (a) What speed do the electrons have when they strike the screen? (b) What is their kinetic energy in joules?
42. Electrons are accelerated to an energy of 20.0 GeV in the 3.00-km-long Stanford Linear Accelerator. (a) What is the γ factor for the electrons? (b) What is their speed? (c) How long does the accelerator appear to them?
43. A pion at rest ($m_\pi = 270m_e$) decays to a muon ($m_\mu = 206m_e$) and an antineutrino ($m_{\bar{\nu}} \approx 0$). The reaction is written $\pi^- \rightarrow \mu^- + \bar{\nu}$. Find the kinetic energy of the muon and the antineutrino in electron volts. (*Hint:* Relativistic momentum is conserved.)

Section 39.8 Equivalence of Mass and Energy

44. Make an order-of-magnitude estimate of the ratio of mass increase to the original mass of a flag as you run it up a flagpole. In your solution explain what quantities you take as data and the values you estimate or measure for them.
45. When 1.00 g of hydrogen combines with 8.00 g of oxygen, 9.00 g of water is formed. During this chemical reaction, 2.86×10^5 J of energy is released. How much mass do the constituents of this reaction lose? Is the loss of mass likely to be detectable?
46. A spaceship of mass 1.00×10^6 kg is to be accelerated to $0.600c$. (a) How much energy does this require? (b) How many kilograms of matter would it take to provide this much energy?
47. In a nuclear power plant the fuel rods last 3 yr before they are replaced. If a plant with rated thermal power 1.00 GW operates at 80.0% capacity for the 3 yr, what is the loss of mass of the fuel?
48. A ^{57}Fe nucleus at rest emits a 14.0-keV photon. Use the conservation of energy and momentum to deduce the kinetic energy of the recoiling nucleus in electron volts. (Use $Mc^2 = 8.60 \times 10^{-9}$ J for the final state of the ^{57}Fe nucleus.)
49. The power output of the Sun is 3.77×10^{26} W. How much mass is converted to energy in the Sun each second?
50. A gamma ray (a high-energy photon of light) can produce an electron (e^-) and a positron (e^+) when

it enters the electric field of a heavy nucleus: $\gamma \rightarrow e^+ + e^-$. What minimum γ-ray energy is required to accomplish this task? (*Hint:* The masses of the electron and the positron are equal.)

Section 39.9 Relativity and Electromagnetism

51. As measured by observers in a reference frame S, a particle having charge q moves with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$. The resulting force on the particle is then measured to be $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Another observer moves along with the charged particle and also measures its charge to be q but measures the electric field to be $\mathbf{E}'$. If both observers are to measure the same force $\mathbf{F}$, show that $\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}$.

ADDITIONAL PROBLEMS

52. An electron has a speed of $0.750c$. Find the speed of a proton that has (a) the same kinetic energy as the electron; (b) the same momentum as the electron.

WEB 53. The cosmic rays of highest energy are protons, which have kinetic energy on the order of 10^{13} MeV. (a) How long would it take a proton of this energy to travel across the Milky Way galaxy, having a diameter of $\sim 10^5$ ly, as measured in the proton's frame? (b) From the point of view of the proton, how many kilometers across is the galaxy?

54. A spaceship moves away from the Earth at $0.500c$ and fires a shuttle craft in the forward direction at $0.500c$ relative to the ship. The pilot of the shuttle craft launches a probe at forward speed $0.500c$ relative to the shuttle craft. Determine (a) the speed of the shuttle craft relative to the Earth and (b) the speed of the probe relative to the Earth.

55. The net nuclear fusion reaction inside the Sun can be written as $4\,^1\text{H} \rightarrow {}^4\text{He} + \Delta E$. If the rest energy of each hydrogen atom is 938.78 MeV and the rest energy of the helium-4 atom is 3 728.4 MeV, what is the percentage of the starting mass that is released as energy?

56. An astronaut wishes to visit the Andromeda galaxy (2.00 million lightyears away), making a one-way trip that will take 30.0 yr in the spaceship's frame of reference. If his speed is constant, how fast must he travel relative to the Earth?

57. An alien spaceship traveling at $0.600c$ toward the Earth launches a landing craft with an advance guard of purchasing agents. The lander travels in the same direction with a velocity $0.800c$ relative to the spaceship. As observed on the Earth, the spaceship is 0.200 ly from the Earth when the lander is launched. (a) With what velocity is the lander observed to be approaching by observers on the Earth? (b) What is the distance to the Earth at the time of lander launch, as observed by the aliens? (c) How long does it take the lander to reach the Earth as observed by the aliens on the mother ship? (d) If the lander has a mass of 4.00×10^5 kg, what is its kinetic energy as observed in the Earth reference frame?

58. A physics professor on the Earth gives an exam to her students, who are on a rocket ship traveling at speed v relative to the Earth. The moment the ship passes the professor, she signals the start of the exam. She wishes her students to have time T_0 (rocket time) to complete the exam. Show that she should wait a time (Earth time) of

$$T = T_0\sqrt{\frac{1 - v/c}{1 + v/c}}$$

before sending a light signal telling them to stop. (*Hint:* Remember that it takes some time for the second light signal to travel from the professor to the students.)

59. Spaceship I, which contains students taking a physics exam, approaches the Earth with a speed of $0.600c$ (relative to the Earth), while spaceship II, which contains professors proctoring the exam, moves at $0.280c$ (relative to the Earth) directly toward the students. If the professors stop the exam after 50.0 min have passed on their clock, how long does the exam last as measured by (a) the students? (b) an observer on the Earth?

60. Energy reaches the upper atmosphere of the Earth from the Sun at the rate of 1.79×10^{17} W. If all of this energy were absorbed by the Earth and not re-emitted, how much would the mass of the Earth increase in 1 yr?

61. A supertrain (proper length, 100 m) travels at a speed of $0.950c$ as it passes through a tunnel (proper length, 50.0 m). As seen by a trackside observer, is the train ever completely within the tunnel? If so, with how much space to spare?

62. Imagine that the entire Sun collapses to a sphere of radius R_g such that the work required to remove a small mass m from the surface would be equal to its rest energy mc^2. This radius is called the *gravitational radius* for the Sun. Find R_g. (It is believed that the ultimate fate of very massive stars is to collapse beyond their gravitational radii into black holes.)

63. A charged particle moves along a straight line in a uniform electric field $\mathbf{E}$ with a speed of u. If the motion and the electric field are both in the x direction, (a) show that the acceleration of the charge q in the x direction is given by

$$a = \frac{du}{dt} = \frac{qE}{m}\left(1 - \frac{u^2}{c^2}\right)^{3/2}$$

(b) Discuss the significance of the dependence of the acceleration on the speed. (c) If the particle starts from rest at $x = 0$ at $t = 0$, how would you proceed to find the speed of the particle and its position after a time t has elapsed?

64. (a) Show that the Doppler shift $\Delta\lambda$ in the wavelength of light is described by the expression

$$\frac{\Delta\lambda}{\lambda} + 1 = \sqrt{\frac{c - v}{c + v}}$$

where λ is the source wavelength and v is the speed of relative approach between source and observer. (b) How fast would a motorist have to be going for a red light to appear green? Take 650 nm as a typical wavelength for red light, and one of 550 nm as typical for green.

65. A rocket moves toward a mirror at $0.800c$ relative to the reference frame S in Figure P39.65. The mirror is stationary relative to S. A light pulse emitted by the rocket travels toward the mirror and is reflected back to the rocket. The front of the rocket is 1.80×10^{12} m from the mirror (as measured by observers in S) at the moment the light pulse leaves the rocket. What is the total travel time of the pulse as measured by observers in (a) the S frame and (b) the front of the rocket?

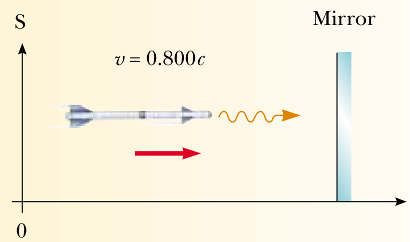


Figure P39.65 Problems 65 and 66.

66. An observer in a rocket moves toward a mirror at speed v relative to the reference frame labeled by S in Figure P39.65. The mirror is stationary with respect to S. A light pulse emitted by the rocket travels toward the mirror and is reflected back to the rocket. The front of the rocket is a distance d from the mirror (as measured by observers in S) at the moment the light pulse leaves the rocket. What is the total travel time of the pulse as measured by observers in (a) the S frame and (b) the front of the rocket?

67. Ted and Mary are playing a game of catch in frame S′, which is moving at $0.600c$, while Jim in frame S watches the action (Fig. P39.67). Ted throws the ball to Mary at $0.800c$ (according to Ted) and their separation (measured in S′) is 1.80×10^{12} m. (a) According to Mary, how fast is the ball moving? (b) According to Mary, how long does it take the ball to reach her? (c) According to Jim, how far apart are Ted and Mary, and how fast is the ball moving? (d) According to Jim, how long does it take the ball to reach Mary?

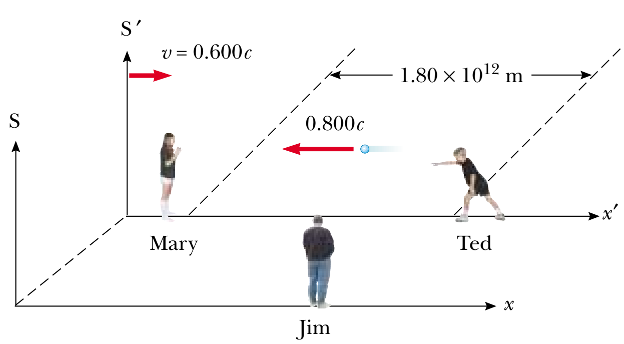


Figure P39.67

68. A rod of length L_0 moving with a speed v along the horizontal direction makes an angle θ_0 with respect to the x' axis. (a) Show that the length of the rod as measured by a stationary observer is $L = L_0[1 - (v^2/c^2)\cos^2\theta_0]^{1/2}$. (b) Show that the angle that the rod makes with the x axis is given by $\tan\theta = \gamma\tan\theta_0$. These results show that the rod is both contracted and rotated. (Take the lower end of the rod to be at the origin of the primed coordinate system.)

69. Consider two inertial reference frames S and S′, where S′ is moving to the right with a constant speed of $0.600c$ as measured by an observer in S. A stick of proper length 1.00 m moves to the left toward the origins of both S and S′, and the length of the stick is 50.0 cm as measured by an observer in S′. (a) Determine the speed of the stick as measured by observers in S and S′. (b) What is the length of the stick as measured by an observer in S?

70. Suppose our Sun is about to explode. In an effort to escape, we depart in a spaceship at $v = 0.800c$ and head toward the star Tau Ceti, 12.0 ly away. When we reach the midpoint of our journey from the Earth, we see our Sun explode and, unfortunately, at the same instant we see Tau Ceti explode as well. (a) In the spaceship's frame of reference, should we conclude that the two explosions occurred simultaneously? If not, which occurred first? (b) In a frame of reference in which the Sun and Tau Ceti are at rest, did they explode simultaneously? If not, which exploded first?

71. The light emitted by a galaxy shows a continuous distribution of wavelengths because the galaxy is composed of billions of different stars and other thermal emitters. Nevertheless, some narrow gaps occur in the continuous spectrum where light has been absorbed by cooler gases in the outer photospheres of normal stars. In particular, ionized calcium atoms at rest produce strong absorption at a wavelength of 394 nm. For a galaxy in the constellation Hydra, 2 billion lightyears away, this absorption line is shifted to 475 nm. How fast is the galaxy moving away from the Earth? (*Note:* The assumption that the recession speed is small compared with c, as made in Problem 21, is not a good approximation here.)

72. Prepare a graph of the relativistic kinetic energy and the classical kinetic energy, both as a function of speed, for an object with a mass of your choice. At what speed does the classical kinetic energy underestimate the relativistic value by 1 percent? By 5 percent? By 50 percent?

73. The total volume of water in the oceans is approximately $1.40 \times 10^9\ \text{km}^3$. The density of sea water is $1\,030\ \text{kg/m}^3$, and the specific heat of the water is $4\,186\ \text{J/(kg}\cdot{}^\circ\text{C)}$. Find the increase in mass of the oceans produced by an increase in temperature of 10.0°C.

Answers to Quick Quizzes

39.1 They both are because they can report only what they see. They agree that the person in the truck throws the ball up and then catches it a bit later.

39.2 It depends on the direction of the throw. Taking the direction in which the train is traveling as the positive x direction, use the values $u_x' = +90$ mi/h and $v = +110$ mi/h, with u_x in Equation 39.2 being the value you are looking for. If the pitcher throws the ball in the same direction as the train, a person at rest on the Earth sees the ball moving at 110 mi/h + 90 mi/h = 200 mi/h. If the pitcher throws in the opposite direction, the person on the Earth sees the ball moving in the same direction as the train but at only 110 mi/h − 90 mi/h = 20 mi/h.

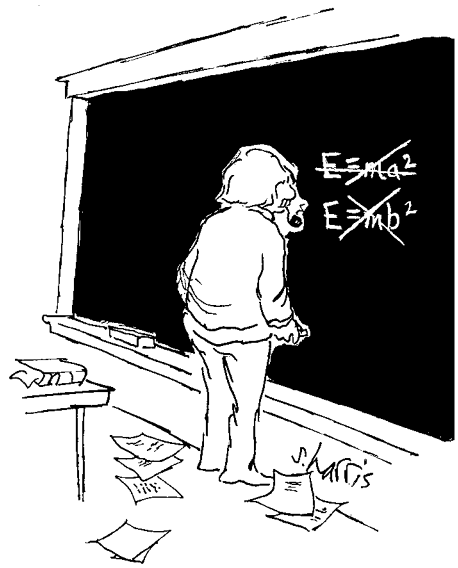
39.3 Both are correct. Although the two observers reach different conclusions, each is correct in her or his own reference frame because the concept of simultaneity is not absolute.

39.4 About 2.9×10^8 m/s, because this is the speed at which $\gamma = 5$. For every 5 s ticking by on the Mission Control clock, the Earth-bound observer (with a powerful telescope!) sees the rocket clock ticking off 1 s. The astronaut sees her own clock operating at a normal rate. To her, Mission Control is moving away from her at a speed of 2.9×10^8 m/s, and she sees the Mission Control clock as running slow. Strange stuff, this relativity!

39.5 If their on-duty time is based on clocks that remain on the Earth, they will have larger paychecks. Less time will have passed for the astronauts in their frame of reference than for their employer back on the Earth.

39.6 By a curved line. This can be seen in the middle of Speedo's world-line in Figure 39.14, where he turns around and begins his trip home.

39.7 The downstairs clock runs more slowly because it is closer to the Earth and hence experiences a stronger gravitational field than the upstairs clock does.



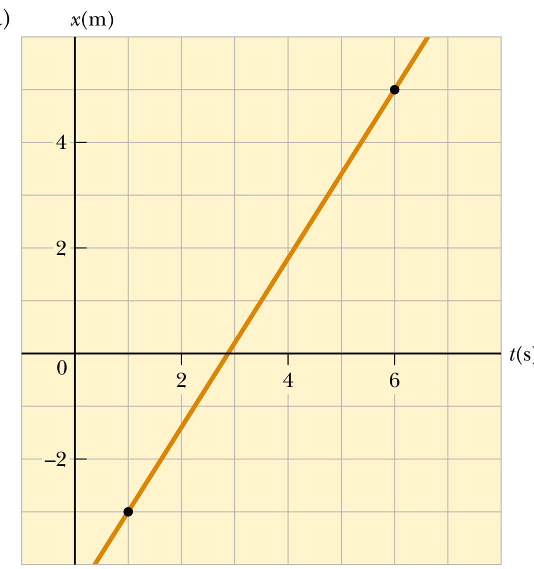
Answers to Odd-Numbered Problems

Chapter 1

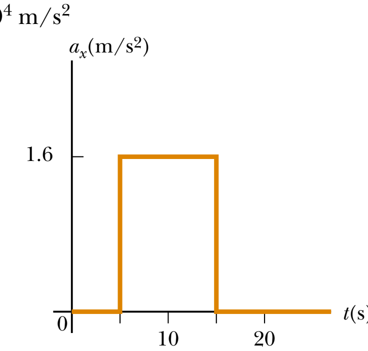
1. 2.15×10^4 kg/m^3
3. 184 g
5. (a) 7.10 cm^3 (b) 1.18×10^{-29} m^3 (c) 0.228 nm (d) 12.7 cm^3, 2.11×10^{-29} m^3, 0.277 nm
7. (a) 4.00 u = 6.64×10^{-24} g (b) 55.9 u = 9.29×10^{-23} g (c) 207 u = 3.44×10^{-22} g
9. (a) 9.83×10^{-16} g (b) 1.06×10^7 atoms
11. (a) 4.01×10^{25} molecules (b) 3.65×10^4 molecules
13. no
15. (b) only
17. $0.579t$ ft^3/s + $1.19 \times 10^{-9}t^2$ ft^3/s^2
19. 1.39×10^3 m^2
21. (a) 0.071 4 gal/s (b) 2.70×10^{-4} m^3/s (c) 1.03 h
23. 4.05×10^3 m^2
25. 11.4×10^3 kg/m^3
27. 1.19×10^{57} atoms
29. (a) 190 y (b) 2.32×10^4 times
31. 151 μm
33. 1.00×10^{10} lb
35. 3.08×10^4 m^3
37. 5.0 m
39. 2.86 cm
41. $\sim 10^6$ balls
43. $\sim 10^7$ or 10^8 rev
45. $\sim 10^7$ or 10^8 blades
47. $\sim 10^2$ kg; $\sim 10^3$ kg
49. $\sim 10^2$ tuners
51. (a) (346 ± 13) m^2 (b) (66.0 ± 1.3) m
53. $(1.61 \pm 0.17) \times 10^3$ kg/m^3
55. 115.9 m
57. 316 m
59. 4.50 m^2
61. 3.41 m
63. 0.449%
65. (a) 0.529 cm/s (b) 11.5 cm/s
67. 1×10^{10} gal/yr
69. $\sim 10^{11}$ stars

Chapter 2

1. (a) 2.30 m/s (b) 16.1 m/s (c) 11.5 m/s
3. (a) 5 m/s (b) 1.2 m/s (c) -2.5 m/s (d) -3.3 m/s (e) 0
5. (a) 3.75 m/s (b) 0
7. (a)



(b) 1.60 m/s
9. (a) -2.4 m/s (b) -3.8 m/s (c) 4.0 s
11. (a) 5.0 m/s (b) -2.5 m/s (c) 0 (d) 5.0 m/s
13. 1.34×10^4 m/s^2
15. (a)

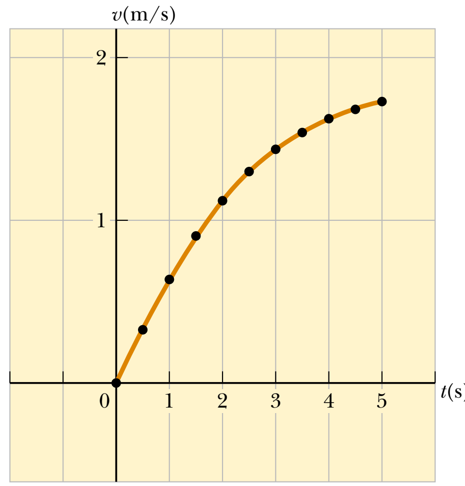


(b) 1.6 m/s^2 and 0.80 m/s^2
17. (a) 2.00 m (b) -3.00 m/s (c) -2.00 m/s^2
19. (a) 1.3 m/s^2 (b) 2.0 m/s^2 at 3 s (c) at $t = 6$ s and for $t > 10$ s (d) -1.5 m/s^2 at 8 s
21. 2.74×10^5 m/s^2, which is 2.79×10^4 g
23. (a) 6.61 m/s (b) -0.448 m/s^2
25. -16.0 cm/s^2
27. (a) 2.56 m (b) -3.00 m/s
29. (a) 8.94 s (b) 89.4 m/s
31. (a) 20.0 s (b) no
33. $x_f - x_i = v_{xf}\,t - a_x t^2/2$; $v_{xf} = 3.10$ m/s

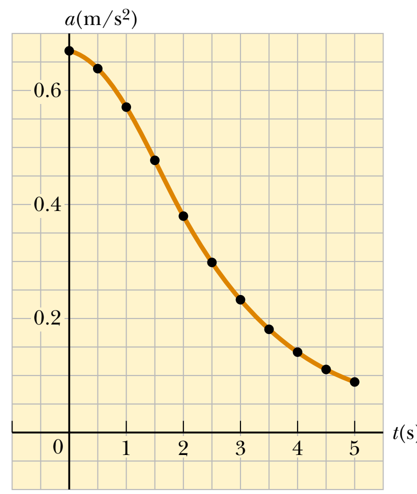
35. (a) 35.0 s (b) 15.7 m/s
37. (a) $-202\ \text{m/s}^2$ (b) 198 m
39. (a) 3.00 m/s (b) 6.00 s (c) $-0.300\ \text{m/s}^2$ (d) 2.05 m/s
41. (a) -4.90 m, -19.6 m, -44.1 m (b) -9.80 m/s, -19.6 m/s, -29.4 m/s
43. (a) 10.0 m/s up (b) 4.68 m/s down
45. No. In 0.2 s the bill falls out from between David's fingers.
47. (a) 29.4 m/s (b) 44.1 m
49. (a) 7.82 m (b) 0.782 s
51. (a) 1.53 s (b) 11.5 m (c) -4.60 m/s, $-9.80\ \text{m/s}^2$
53. (a) $a_x = a_{xi} + Jt$, $v_x = v_{xi} + a_{xi}t + \frac{1}{2}Jt^2$, $x = x_i + v_{xi}t + \frac{1}{2}a_{xi}t^2 + \frac{1}{6}Jt^3$
55. 0.222 s
57. 0.509 s
59. (a) 41.0 s (b) 1.73 km (c) -184 m/s
61. $v_{xi}\,t + at^2/2$, in agreement with Equation 2.11
63. (a) $5.43\ \text{m/s}^2$ and $3.83\ \text{m/s}^2$ (b) 10.9 m/s and 11.5 m/s (c) Maggie by 2.62 m
65. (a) 45.7 s (b) 574 m (c) 12.6 m/s (d) 765 s
67. (a) 2.99 s (b) -15.4 m/s (c) 31.3 m/s down and 34.9 m/s down
69. (a) 5.46 s (b) 73.0 m (c) $v_{\text{Stan}} = 22.6$ m/s, $v_{\text{Kathy}} =$ 26.7 m/s
71. (a) See top of next column. (b) See top of next column.
73. $0.577v$

Chapter 3

1. $(-2.75, -4.76)$ m
3. 1.15; 2.31
5. (a) 2.24 m (b) 2.24 m at 26.6° from the positive x axis.
7. (a) 484 m (b) 18.1° north of west
9. 70.0 m
11. (a) approximately 6.1 units at 112° (b) approximately 14.8 units at 22°
13. (a) 10.0 m (b) 15.7 m (c) 0
15. (a) 5.2 m at 60° (b) 3.0 m at 330° (c) 3.0 m at 150° (d) 5.2 m at 300°
17. approximately 420 ft at $-3°$
19. 5.83 m at 59.0° to the right of his initial direction
21. 1.31 km north and 2.81 km east
23. (a) 10.4 cm (b) 35.5°
25. 47.2 units at 122° from the positive x axis.
27. $(-25.0\mathbf{i})$m + $(43.3\mathbf{j})$m
29. 7.21 m at 56.3° from the positive x axis.
31. (a) $2.00\mathbf{i} - 6.00\mathbf{j}$ (b) $4.00\mathbf{i} + 2.00\mathbf{j}$ (c) 6.32 (d) 4.47 (e) 288°; 26.6° from the positive x axis.
33. (a) $(-11.1\mathbf{i} + 6.40\mathbf{j})$ m (b) $(1.65\mathbf{i} + 2.86\mathbf{j})$ cm (c) $(-18.0\mathbf{i} - 12.6\mathbf{j})$ in.
35. 9.48 m at 166°
37. (a) 185 N at 77.8° from the positive x axis (b) $(-39.3\mathbf{i} - 181\mathbf{j})$ N
39. $\mathbf{A} + \mathbf{B} = (2.60\mathbf{i} + 4.50\mathbf{j})$ m



Chapter 2, Problem 71(a)

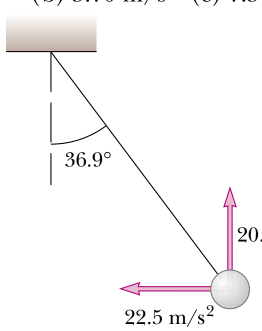


Chapter 2, Problem 71(b)

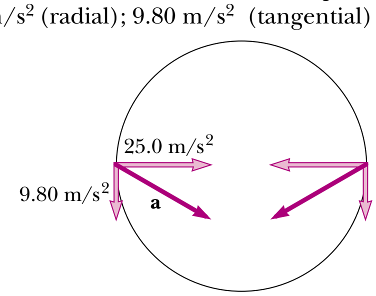
41. 196 cm at $-14.7°$ from the positive x axis.
43. (a) $8.00\mathbf{i} + 12.0\mathbf{j} - 4.00\mathbf{k}$ (b) $2.00\mathbf{i} + 3.00\mathbf{j} - 1.00\mathbf{k}$ (c) $-24.0\mathbf{i} - 36.0\mathbf{j} + 12.0\mathbf{k}$
45. (a) 5.92 m is the magnitude of $(5.00\mathbf{i} - 1.00\mathbf{j} - 3.00\mathbf{k})$ m (b) 19.0 m is the magnitude of $(4.00\mathbf{i} - 11.0\mathbf{j} + 15.0\mathbf{k})$ m
47. 157 km
49. (a) $-3.00\mathbf{i} + 2.00\mathbf{j}$ (b) 3.61 at 146° from the positive x axis. (c) $3.00\mathbf{i} - 6.00\mathbf{j}$
51. (a) $49.5\mathbf{i} + 27.1\mathbf{j}$ (b) 56.4 units at 28.7° from the positive x axis.
53. 1.15°
55. (a) 2.00, 1.00, 3.00 (b) 3.74 (c) $\theta_x = 57.7°$, $\theta_y = 74.5°$, $\theta_z = 36.7°$
57. 240 m at 237°
59. 390 mi/h at 7.37° north of east
61. $\mathbf{R}_1 = a\mathbf{i} + b\mathbf{j}$; $R_1 = \sqrt{a^2 + b^2}$ (b) $\mathbf{R}_2 = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

Chapter 4

1. (a) 4.87 km at 209° from east (b) 23.3 m/s (c) 13.5 m/s at 209°
3. (a) $(18.0t)\mathbf{i} + (4.00t - 4.90t^2)\mathbf{j}$ (b) $18.0\mathbf{i} + (4.00 - 9.80t)\mathbf{j}$ (c) $-9.80\mathbf{j}$ (d) $(54.0\mathbf{i} - 32.1\mathbf{j})$ m (e) $(18.0\mathbf{i} - 25.4\mathbf{j})$ m/s (f) $(-9.80\mathbf{j})$ m/s^2
5. (a) $(2.00\mathbf{i} + 3.00\mathbf{j})$ m/s^2 (b) $(3.00t + t^2)\mathbf{i}$ m, $(1.50\,t^2 - 2.00t)\mathbf{j}$ m
7. (a) $(0.800\mathbf{i} - 0.300\mathbf{j})$ m/s^2 (b) 339° (c) $(360\mathbf{i} - 72.7\mathbf{j})$ m, $-15.2°$
9. (a) $(3.34\mathbf{i})$ m/s (b) $-50.9°$
11. (a) 20.0° (b) 3.05 s
13. $x = 7.23$ km $y = 1.68$ km
15. 53.1°
17. 22.4° or 89.4°
19. (a) The ball clears by 0.889 m (b) while descending
21. $d\tan\theta_i - gd^2/(2v_i^2\cos^2\theta_i)$
23. (a) 0.852 s (b) 3.29 m/s (c) 4.03 m/s (d) 50.8° (e) 1.12 s
25. 377 m/s^2
27. 10.5 m/s, 219 m/s^2
29. (a) 6.00 rev/s (b) 1.52 km/s^2 (c) 1.28 km/s^2
31. 1.48 m/s^2 inward at 29.9° behind the radius
33. (a) 13.0 m/s^2 (b) 5.70 m/s (c) 7.50 m/s^2
35. (a)



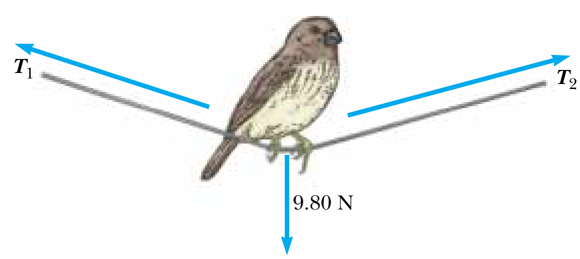
(b) 29.7 m/s^2 (c) 6.67 m/s at 36.9° above the horizontal
37. 2.02×10^3 s; 21.0% longer
39. 153 km/h at 11.3° north of west
41. (a) 36.9° (b) 41.6° (c) 3.00 min
43. 15.3 m
45. $2\,v_i t\cos\theta_i$
47. (b) $45° + \phi/2$; $v_i^2(1 - \sin\phi)/g\cos^2\phi$
49. (a) 41.7 m/s (b) 3.81 s (c) $(34.1\mathbf{i} - 13.4\mathbf{j})$ m/s; 36.6 m/s
51. (a) 25.0 m/s^2 (radial); 9.80 m/s^2 (tangential) (b)



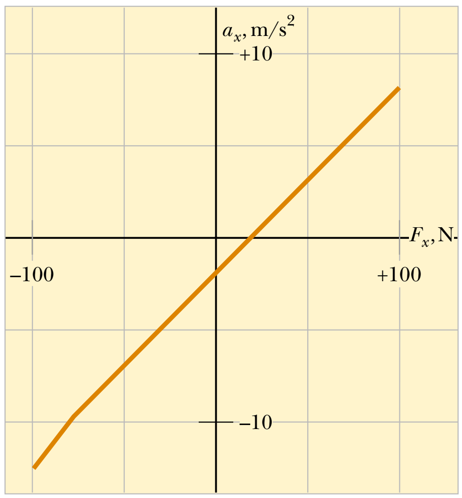
(c) 26.8 m/s^2 inward at 21.4° below the horizontal
53. 8.94 m/s at $-63.4°$ relative to the positive x axis.
55. 20.0 m
57. (a) 0.600 m (b) 0.402 m (c) 1.87 m/s^2 toward center (d) 9.80 m/s^2 down
59. (a) 6.80 km (b) 3.00 km vertically above the impact point (c) 66.2°
61. (a) 46.5 m/s (b) $-77.6°$ (c) 6.34 s
63. (a) 1.53 km (b) 36.2 s (c) 4.04 km
65. (a) 20.0 m/s, 5.00 s (b) $(16.0\mathbf{i} - 27.1\mathbf{j})$ m/s (c) 6.54 s (d) $(24.6\mathbf{i})$ m
67. (a) 43.2 m (b) $(9.66\mathbf{i} - 25.5\mathbf{j})$ m/s
69. Imagine you are shaking down the mercury in a fever thermometer. Starting with your hand at the level of your shoulder, move your hand down as fast as you can and snap it around an arc at the bottom. ~ 100 m/s$^2 \approx 10\,g$

Chapter 5

1. (a) 1/3 (b) 0.750 m/s^2
3. $(6.00\mathbf{i} + 15.0\mathbf{j})$ N; 16.2 N
5. 312 N
7. (a) $x = vt/2$ (b) $F_g v\mathbf{i}/gt + F_g\mathbf{j}$
9. (a) $(2.50\mathbf{i} + 5.00\mathbf{j})$ N (b) 5.59 N
11. (a) 3.64×10^{-18} N (b) 8.93×10^{-30} N is 408 billion times smaller.
13. 2.38 kN
15. (a) 5.00 m/s^2 at 36.9° (b) 6.08 m/s^2 at 25.3°
17. (a) $\sim 10^{-22}$ m/s^2 (b) $\sim 10^{-23}$ m
19. (a) 0.200 m/s^2 forward (b) 10.0 m (c) 2.00 m/s
21. (a) 15.0 lb up (b) 5.00 lb up (c) 0
23. 613 N

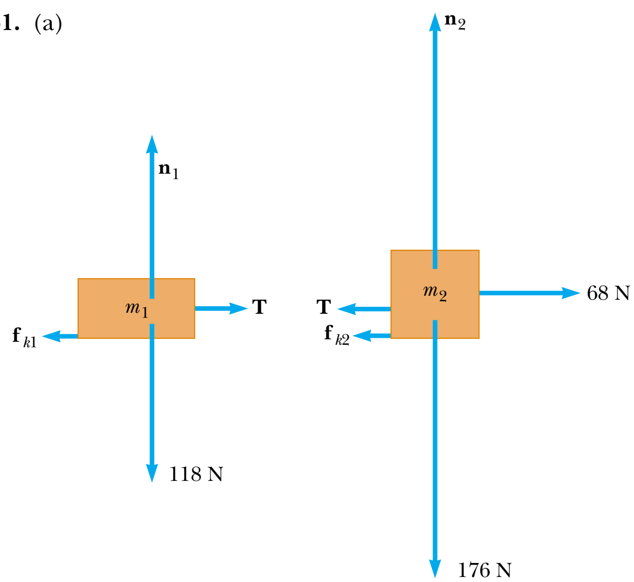


27. (a) 49.0 N (b) 98.0 N (c) 24.5 N
29. 8.66 N east
31. 100 N and 204 N
33. 3.73 m
35. $a = F/(m_1 + m_2)$; $T = F\,m_1/(m_1 + m_2)$
37. (a) $F_x > 19.6$ N (b) $F_x \leq -78.4$ N (c) See top of next page.
39. (a) 706 N (b) 814 N (c) 706 N (d) 648 N
41. $\mu_s = 0.306$; $\mu_k = 0.245$
43. (a) 256 m (b) 42.7 m
45. (a) 1.78 m/s^2 (b) 0.368 (c) 9.37 N (d) 2.67 m/s
47. (a) 0.161 (b) 1.01 m/s^2
49. 37.8 N



Chapter 5, Problem 37(c)

51. (a)



(b) 27.2 N, 1.29 m/s^2

53. Any value between 31.7 N and 48.6 N

55. (a) See top of next column.
(b) 0.408 m/s^2 (c) 83.3 N

57. 1.18 kN

59. (a) $Mg/2$, $Mg/2$, $Mg/2$, $3Mg/2$, Mg (b) $Mg/2$

61. (b)

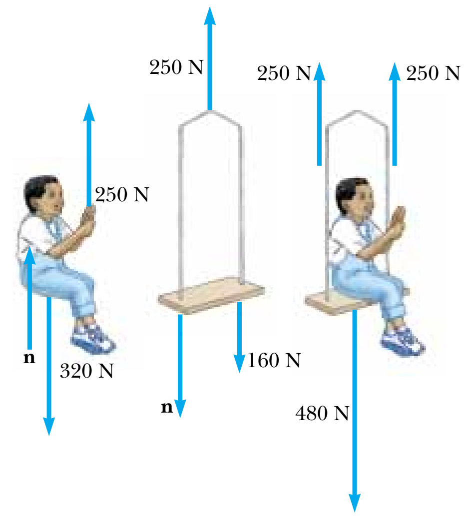
θ	0	15.0°	30.0°	45.0°	60.0°
P(N)	40.0	46.4	60.1	94.3	260

63. (a) 19.3° (b) 4.21 N

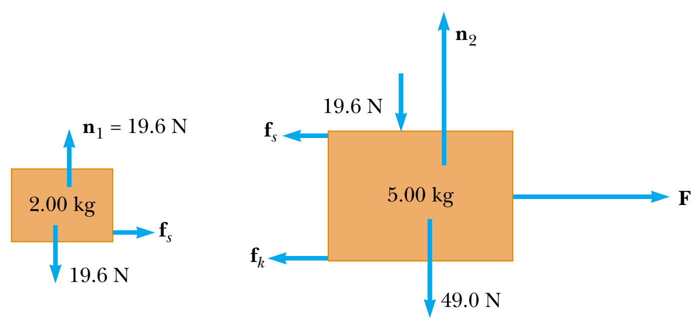
65. (a) 2.13 s (b) 1.67 m

67. (a) See next column.
Static friction between the two blocks accelerates the upper block. (b) 34.7 N (c) 0.306

69. $(M + m_1 + m_2)(m_2 g/m_1)$

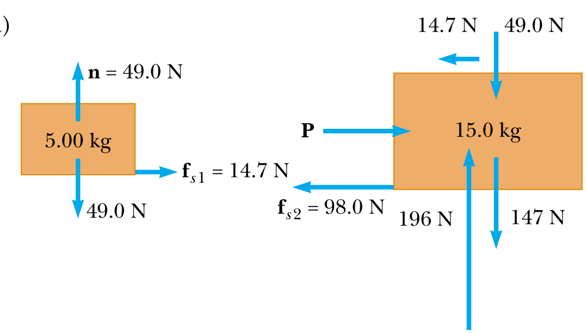


Chapter 5, Problem 55(a)



Chapter 5, Problem 67(a)

71. (a)



(b) 113 N (c) 0.980 m/s^2 and 1.96 m/s^2

73. (a) 0.087 1 (b) 27.4 N

75. (a) 30.7° (b) 0.843 N

77. (a) 3.34 (b) Either the car would flip over backwards, or the wheels would skid, spinning in place, and the time would increase.

Chapter 6

1. (a) 8.00 m/s (b) 3.02 N

3. Any speed up to 8.08 m/s

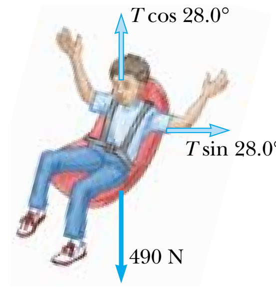
5. 6.22×10^{-12} N
7. (a) 1.52 m/s^2 (b) 1.66 km/s (c) 6 820 s
9. (a) static friction (b) 0.085 0
11. $v \leq 14.3$ m/s
13. (a) 68.6 N toward the center of the circle and 784 N up (b) 0.857 m/s^2
15. No. The jungle lord needs a vine of tensile strength 1.38 kN.
17. (a) 4.81 m/s (b) 700 N up
19. 3.13 m/s
21. (a) 2.49×10^4 N up (b) 12.1 m/s
23. (a) 0.822 m/s^2 (b) 37.0 N (c) 0.0839
25. (a) 17.0° (b) 5.12 N
27. (a) 491 N (b) 50.1 kg (c) 2.00 m/s^2
29. 0.0927°
31. (a) 32.7 s^{-1} (b) 9.80 m/s^2 (c) 4.90 m/s^2
33. 3.01 N
35. (a) 1.47 N·s/m (b) 2.04×10^{-3} s (c) 2.94×10^{-2} N
37. (a) 0.0347 s^{-1} (b) 2.50 m/s (c) $a = -cv$
39. $\sim 10^1$ N
41. (a) 13.7 m/s down
(b)

t (s)	*x* (m)	*v* (m/s)
0	0	0
0.2	0	−1.96
0.4	−0.392	−3.88
. . .	. . .	. . .
1.0	−3.77	−8.71
. . . 2.0	−14.4	−12.56
. . . 4.0	−41.0	−13.67

43. (a) 49.5 m/s and 4.95 m/s
(b)

t (s)	*y* (m)	*v* (m/s)
0	1 000	0
. . . 1	995	−9.7
. . . 2	980	−18.6
. . . 10	674	−47.7
. . . 10.1	671	−16.7
. . . 12	659	−4.95
. . . 145	0	−4.95

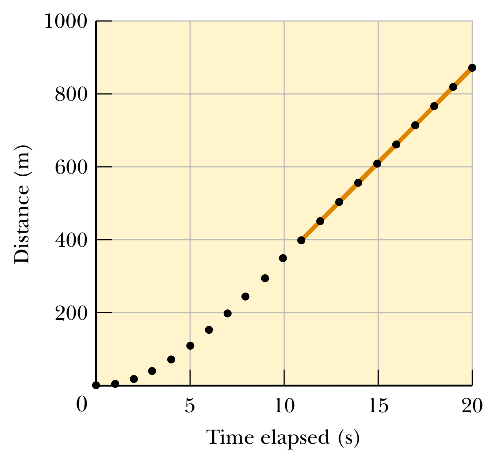
45. (a) 2.33×10^{-4} kg/m (b) 53 m/s (c) 42 m/s. The second trajectory is higher and shorter. In both, the ball attains maximum height when it has covered about 57% of its horizontal range, and it attains minimum speed somewhat later. The impact speeds also are both about 30 m/s.
47. (a) $mg - mv^2/R$ (b) $\sqrt{gR}$
49. (a) 2.63 m/s^2 (b) 201 m (c) 17.7 m/s
51. (a) 9.80 N (b) 9.80 N (c) 6.26 m/s
53. (b) 732 N down at the equator and 735 N down at the poles
59. (a) 1.58 m/s^2 (b) 455 N (c) 329 N (d) 397 N upward and 9.15° inward
61. (a) 5.19 m/s (b) Child + seat:



$T = 555$ N
63. (b) 2.54 s; 23.6 rev/min
65. 215 N horizontally inward
67. (a) either 70.4° or 0° (b) 0°
69. 12.8 N
71. (a)

t (s)	*d* (m)
0	0
1	4.9
2	18.9
. . . 5	112.6
. . . 10	347.0
11	399.1
. . . 15	611.3
. . . 20	876.5

(b)



(c) The graph is straight for 11 s $< t <$ 20 s, with slope 53.0 m/s.

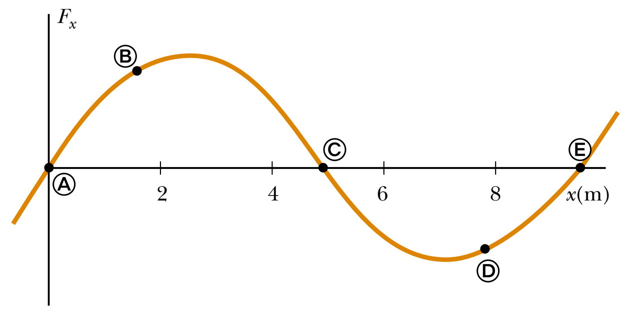
Chapter 7

1. 15.0 MJ
3. (a) 32.8 mJ (b) −32.8 mJ
5. (a) 31.9 J (b) 0 (c) 0 (d) 31.9 J
7. 4.70 kJ
9. 14.0
11. (a) 16.0 J (b) 36.9°
13. (a) 11.3° (b) 156° (c) 82.3°

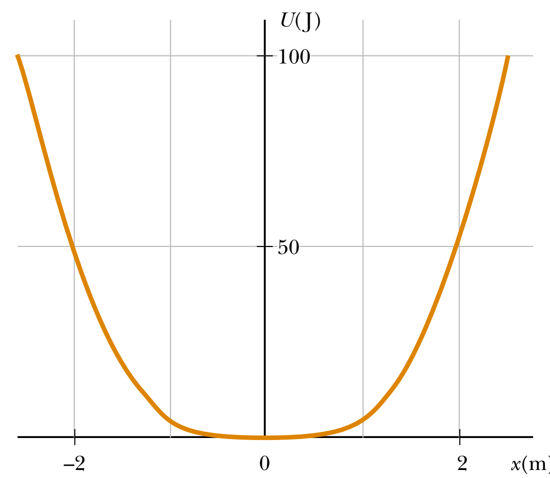
15. (a) 24.0 J (b) -3.00 J (c) 21.0 J
17. (a) 7.50 J (b) 15.0 J (c) 7.50 J (d) 30.0 J
19. (a) 0.938 cm (b) 1.25 J
21. 0.299 m/s
23. 12.0 J
25. (b) mgR
27. (a) 1.20 J (b) 5.00 m/s (c) 6.30 J
29. (a) 60.0 J (b) 60.0 J
31. (a) $\sqrt{2W/m}$ (b) W/d
33. (a) 650 J (b) -588 J (c) 0 (d) 0 (e) 62.0 J (f) 1.76 m/s
35. (a) -168 J (b) -184 J (c) 500 J (d) 148 J (e) 5.64 m/s
37. 2.04 m
39. (a) 22 500 N (b) 1.33×10^{-4} s
41. (a) 0.791 m/s (b) 0.531 m/s
43. 875 W
45. 830 N
47. (a) 5 910 W (b) It is 53.0% of 11 100 W
49. (a) 0.013 5 gal (b) 73.8 (c) 8.08 kW
51. 5.90 km/L
53. (a) 5.37×10^{-11} J (b) 1.33×10^{-9} J
55. 90.0 J
59. (a) $(2 + 24t^2 + 72t^4)$ J (b) $12t$ m/s^2; $48t$ N (c) $(48t + 288t^3)$ W (d) 1 250 J
61. $-0.047\,5$ J
63. 878 kN
65. (b) 240 W
67. (a) $\mathbf{F}_1 = (20.5\mathbf{i} + 14.3\mathbf{j})$ N; $\mathbf{F}_2 = (-36.4\mathbf{i} + 21.0\mathbf{j})$ N (b) $(-15.9\mathbf{i} + 35.3\mathbf{j})$ N (c) $(-3.18\mathbf{i} + 7.07\mathbf{j})$ m/s^2 (d) $(-5.54\mathbf{i} + 23.7\mathbf{j})$ m/s (e) $(-2.30\mathbf{i} + 39.3\mathbf{j})$ m (f) 1 480 J (g) 1 480 J
69. (a) 4.12 m (b) 3.35 m
71. 1.68 m/s
73. (a) 14.5 m/s (b) 1.75 kg (c) 0.350 kg
75. 0.799 J

Chapter 8

1. (a) 259 kJ, 0, -259 kJ (b) 0, -259 kJ, -259 kJ
3. (a) -196 J (b) -196 J (c) -196 J. The force is conservative.
5. (a) 125 J (b) 50.0 J (c) 66.7 J (d) Nonconservative. The results differ.
7. (a) 40.0 J (b) -40.0 J (c) 62.5 J
9. (a) $Ax^2/2 - Bx^3/3$ (b) $\Delta U = 5A/2 - 19B/3$; $\Delta K = -5A/2 + 19B/3$
11. 0.344 m
13. (a) $v_B = 5.94$ m/s; $v_C = 7.67$ m/s (b) 147 J
15. $v = (3gR)^{1/2}$, 0.098 0 N down
17. 10.2 m
19. (a) 19.8 m/s (b) 78.4 J (c) 1.00
21. (a) 4.43 m/s (b) 5.00 m
23. (a) 18.5 km, 51.0 km (b) 10.0 MJ
25. (b) 60.0°
27. 5.49 m/s
29. 2.00 m/s, 2.79 m/s, 3.19 m/s
31. 3.74 m/s
33. (a) -160 J (b) 73.5 J (c) 28.8 N (d) 0.679
35. 489 kJ
37. (a) 1.40 m/s (b) 4.60 cm after release (c) 1.79 m/s
39. 1.96 m
41. (A/r^2) away from the other particle
43. (a) $r = 1.5$ mm, stable; 2.3 mm, unstable; 3.2 mm, stable; $r \rightarrow \infty$ neutral (b) $-5.6\text{ J} < E < 1$ J (c) $0.6\text{ mm} < r < 3.6$ mm (d) 2.6 J (e) 1.5 mm (f) 4 J
45. (a) + at Ⓑ, − at Ⓓ, 0 at Ⓐ, Ⓒ, and Ⓔ (b) Ⓒ stable; Ⓐ and Ⓔ unstable
(c)



47. (b)



Equilibrium at $x = 0$ (c) $v = \sqrt{0.800\text{ J}/m}$
49. (a) 1.50×10^{-10} J (b) 1.07×10^{-9} J (c) 9.15×10^{-10} J
51. 48.2° Note that the answer is independent of the pumpkin's mass and of the radius of the dome.
53. (a) 0.225 J (b) $\Delta E_f = -0.363$ J (c) No; the normal force changes in a complicated way.
55. $\sim 10^2$ W sustainable power
57. 0.327
59. (a) 23.6 cm (b) 5.90 m/s^2 up the incline; no. (c) Gravitational potential energy turns into kinetic energy plus elastic potential energy and then entirely into elastic potential energy.
61. 1.25 m/s

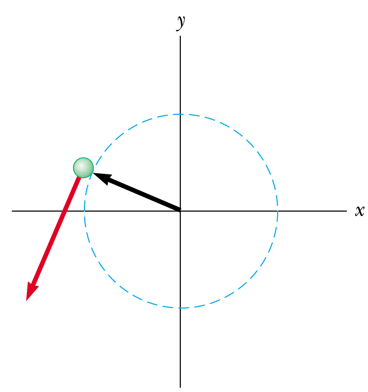
63. (a) 0.400 m (b) 4.10 m/s (c) The block stays on the track.
65. (b) 2.06 m/s
67. (b) 1.44 m (c) 0.400 m (d) No. A very strong wind pulls the string out horizontally (parallel to the ground). The largest possible equilibrium height is equal to L.
71. (a) 6.15 m/s (b) 9.87 m/s
73. 0.923 m/s

Chapter 9

1. (a) $(9.00\mathbf{i} - 12.0\mathbf{j})$ kg·m/s (b) 15.0 kg·m/s at 307°
3. 6.25 cm/s west
5. $\sim 10^{-23}$ m/s
7. (b) $p = \sqrt{2mK}$
9. (a) 13.5 N·s (b) 9.00 kN (c) 18.0 kN
11. 260 N normal to the wall
13. 15.0 N in the direction of the initial velocity of the exiting water stream
15. 65.2 m/s
17. 301 m/s
19. (a) $v_{gx} = 1.15$ m/s (b) $v_{px} = -0.346$ m/s
21. (a) 20.9 m/s east (b) 8.68 kJ into internal energy
23. (a) 2.50 m/s (b) 37.5 kJ (c) Each process is the time-reversal of the other. The same momentum conservation equation describes both.
25. (a) 0.284 (b) 115 fJ and 45.4 fJ
27. 91.2 m/s
29. (a) 2.88 m/s at 32.3° north of east (b) 783 J into internal energy
31. No; his speed was 41.5 mi/h.
33. 2.50 m/s at $-60.0°$
35. $(3.00\mathbf{i} - 1.20\mathbf{j})$ m/s
37. Orange: $v_i \cos\theta$; yellow: $v_i \sin\theta$
39. (a) $(-9.33\mathbf{i} - 8.33\mathbf{j})$ Mm/s (b) 439 fJ
41. $\mathbf{r}_{CM} = (11.7\mathbf{i} + 13.3\mathbf{j})$ cm
43. 0.006 73 nm from the oxygen nucleus along the bisector of the angle
45. (a) 15.9 g (b) 0.153 m
47. 0.700 m
49. (a) $(1.40\mathbf{i} + 2.40\mathbf{j})$ m/s (b) $(7.00\mathbf{i} + 12.0\mathbf{j})$ kg·m/s
51. (a) 39.0 MN up (b) 3.20 m/s^2 up
53. (a) 442 metric tons (b) 19.2 metric tons
55. (a) $(1.33\mathbf{i})$ m/s (b) $(-235\mathbf{i})$ N (c) 0.680 s (d) $(-160\mathbf{i})$ N·s and $(+160\mathbf{i})$ N·s (e) 1.81 m (f) 0.454 m (g) -427 J (h) $+107$ J (i) Equal friction forces act through different distances on person and cart to do different amounts of work on them. The total work on both together, -320 J, becomes $+320$ J of internal energy in this perfectly inelastic collision.
57. 1.39 km/s
59. 240 s
61. 0.980 m
63. (a) 6.81 m/s (b) 1.00 m
65. $(3Mgx/L)\mathbf{j}$
67. (a) 3.75 kg·m/s^2 (b) 3.75 N (c) 3.75 N (d) 2.81 J (e) 1.41 J (f) Friction between sand and belt converts half of the input work into internal energy.
69. (a) As the child walks to the right, the boat moves to the left and the center of mass remains fixed. (b) 5.55 m from the pier (c) No, since 6.55 m is less than 7.00 m.
71. (a) 100 m/s (b) 374 J
73. (a) $\sqrt{2}\, v_i$ for m and $\sqrt{2/3}\, v_i$ for $3m$ (b) 35.3°
75. (a) 3.73 km/s (b) 153 km

Chapter 10

1. (a) 4.00 rad/s^2 (b) 18.0 rad
3. (a) 1 200 rad/s (b) 25.0 s
5. (a) 5.24 s (b) 27.4 rad
7. (a) 5.00 rad, 10.0 rad/s, 4.00 rad/s^2 (b) 53.0 rad, 22.0 rad/s, 4.00 rad/s^2
9. 13.7 rad/s^2
11. $\sim 10^7$ rev/y
13. (a) 0.180 rad/s (b) 8.10 m/s^2 toward the center of the track
15. (a) 8.00 rad/s (b) 8.00 m/s, $a_r = -64.0$ m/s^2, $a_t = 4.00$ m/s^2 (c) 9.00 rad
17. (a) 54.3 rev (b) 12.1 rev/s
19. (a) 126 rad/s (b) 3.78 m/s (c) 1.27 km/s^2 (d) 20.2 m
21. (a) $-2.73\mathbf{i}$ m $+ 1.24\mathbf{j}$ m (b) second quadrant, 156° (c) $-1.85\mathbf{i}$ m/s $- 4.10\mathbf{j}$ m/s (d) into the third quadrant at 246°



(e) $6.15\mathbf{i}$ m/s^2 $- 2.78\mathbf{j}$ m/s^2 (f) $24.6\mathbf{i}$ N $- 11.1\mathbf{j}$ N
23. (a) 92.0 kg·m^2, 184 J (b) 6.00 m/s, 4.00 m/s, 8.00 m/s, 184 J
25. (a) 143 kg·m^2 (b) 2.57 kJ
29. 1.28 kg·m^2
31. $\sim 10^0 = 1$ kg·m^2
33. -3.55 N·m
35. 882 N·m
37. (a) 24.0 N·m (b) 0.035 6 rad/s^2 (c) 1.07 m/s^2
39. (a) 0.309 m/s^2 (b) 7.67 N and 9.22 N
41. (a) 872 N (b) 1.40 kN

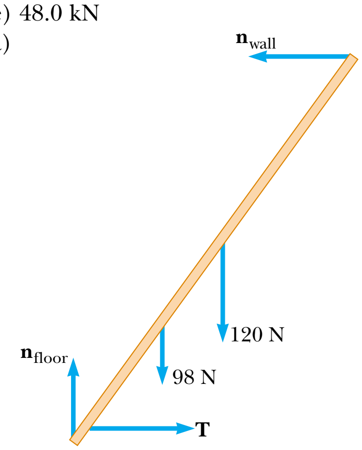
43. 2.36 m/s
45. (a) 11.4 N, 7.57 m/s^2, 9.53 m/s down (b) 9.53 m/s
49. (a) $2(Rg/3)^{1/2}$ (b) $4(Rg/3)^{1/2}$ (c) $(Rg)^{1/2}$
51. $\frac{1}{3}\ell$
53. (a) 1.03 s (b) 10.3 rev
55. (a) 4.00 J (b) 1.60 s (c) yes
57. (a) 12.5 rad/s (b) 128 rad
59. (a) $(3g/L)^{1/2}$ (b) $3g/2L$ (c) $-\frac{3}{2}g\mathbf{i} - \frac{3}{4}g\mathbf{j}$
(d) $-\frac{3}{2}Mg\mathbf{i} + \frac{1}{4}Mg\mathbf{j}$
61. $\alpha = g(h_2 - h_1)/2\pi R^2$
63. (b) $2gM(\sin\theta - \mu\cos\theta)(m + 2M)^{-1}$
65. 139 m/s
67. 5.80 kg·m^2; the height makes no difference.
69. (a) 2 160 N·m (b) 439 W
71. (a) 118 N and 156 N (b) 1.19 kg·m^2
73. (a) $\alpha = -0.176$ rad/s^2 (b) 1.29 rev (c) 9.26 rev

Chapter 11

1. (a) 500 J (b) 250 J (c) 750 J
3. $\frac{7}{10}Mv^2$
5. (a) $\frac{2}{3}g\sin\theta$ for the disk, larger than $\frac{1}{2}g\sin\theta$ for the hoop
(b) $\frac{1}{3}\tan\theta$
7. 1.21×10^{-4} kg·m^2. The height is unnecessary.
9. $-7.00\mathbf{i} + 16.0\mathbf{j} - 10.0\mathbf{k}$
11. (a) $-17.0\mathbf{k}$ (b) 70.5°
13. (a) 2.00 N·m (b) $\mathbf{k}$
15. (a) negative z direction (b) positive z direction
17. 45.0°
19. $(17.5\mathbf{k})$ kg·m^2/s
21. $(60.0\mathbf{k})$ kg·m^2/s
23. $mvR[\cos(vt/R) + 1]\mathbf{k}$
25. (a) zero (b) $(-mv_i^3\sin^2\theta\cos\theta/2g)\mathbf{k}$
(c) $(-2mv_i^3\sin^2\theta\cos\theta/g)\mathbf{k}$ (d) The downward force of gravity exerts a torque in the $-z$ direction.
27. $-m\ell gt\cos\theta\,\mathbf{k}$
29. 4.50 kg·m^2/s up
31. (a) 0.433 kg·m^2/s (b) 1.73 kg·m^2/s
33. (a) $\omega_f = \omega_i I_1/(I_1 + I_2)$ (b) $I_1/(I_1 + I_2)$
35. (a) 1.91 rad/s (b) 2.53 J, 6.44 J
37. (a) 0.360 rad/s counterclockwise (b) 99.9 J
39. (a) $mv\ell$ down (b) $M/(M + m)$
41. (a) $\omega = 2mv_id/(M + 2m)R^2$ (b) No; some mechanical energy changes into internal energy.
43. (a) 2.19×10^6 m/s (b) 2.18×10^{-18} J
(c) 4.13×10^{16} rad/s
45. $[10Rg(1 - \cos\theta)/7r^2]^{1/2}$
51. (a) $2.70R$ (b) $F_x = -\frac{20}{7}mg$, $F_y = -mg$
53. 0.632
55. (a) v_ir_i/r (b) $T = (mv_i^2r_i^2)r^{-3}$ (c) $\frac{1}{2}mv_i^2(r_i^2/r^2 - 1)$
(d) 4.50 m/s, 10.1 N, 0.450 J
57. 54.0°
59. (a) 3 750 kg·m^2/s (b) 1.88 kJ (c) 3 750 kg·m^2/s
(d) 10.0 m/s (e) 7.50 kJ (f) 5.62 kJ
61. $(M/m)[3ga(\sqrt{2} - 1)]^{1/2}$
63. (c) $(8Fd/3M)^{1/2}$
67. (a) 0.800 m/s^2, 0.400 m/s^2 (b) 0.600 N backward on the plank and forward on the roller, at the top of each roller; 0.200 N forward on each roller and backward on the floor, at the bottom of each roller.

Chapter 12

1. 10.0 N up; 6.00 N·m counterclockwise
3. $[(m_1 + m_b)d + m_1\ell/2]/m_2$
5. -0.429 m
7. (3.85 cm, 6.85 cm)
9. (-1.50 m, -1.50 m)
11. (a) 859 N (b) 1 040 N left and upward at 36.9°
13. (a) $f_s = 268$ N, $n = 1\,300$ N (b) 0.324
15. (a) 1.04 kN at 60.0° (b) $(370\mathbf{i} + 900\mathbf{j})$ N
17. 2.94 kN on each rear wheel and 4.41 kN on each front wheel
19. (a) 29.9 N (b) 22.2 N
21. (a) 35.5 kN (b) 11.5 kN (c) -4.19 kN
23. 88.2 N and 58.8 N
25. 4.90 mm
27. 0.023 8 mm
29. 0.912 mm
31. $\dfrac{8m_1m_2gL_i}{\pi d^2Y(m_1 + m_2)}$
33. (a) 3.14×10^4 N (b) 6.28×10^4 N
35. 1.80×10^8 N/m^2
37. $n_A = 5.98 \times 10^5$ N, $n_B = 4.80 \times 10^5$ N
39. (a) 0.400 mm (b) 40.0 kN (c) 2.00 mm (d) 2.40 mm
(e) 48.0 kN
41. (a)

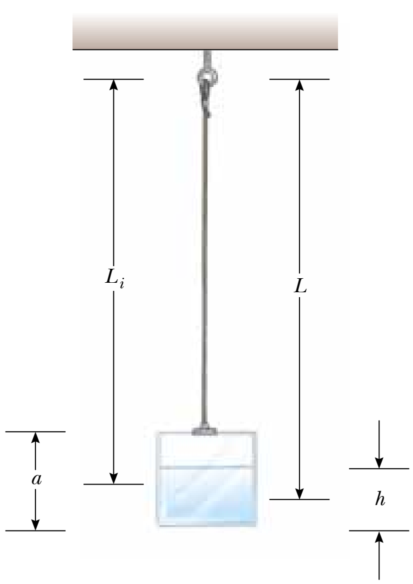


(b) 69.8 N (c) $0.877L$
43. (a) 160 N right (b) 13.2 N right (c) 292 N up
(d) 192 N
45. (a) $T = F_g(L + d)/\sin\theta(2L + d)$
(b) $R_x = F_g(L + d)\cot\theta/(2L + d)$; $R_y = F_gL/(2L + d)$
47. $0.789\,L$
49. 5.08 kN, $R_x = 4.77$ kN, $R_y = 8.26$ kN
51. $T = 2.71$ kN, $R_x = 2.65$ kN
53. (a) $\mu_k = 0.571$; the normal force acts 20.1 cm to the left of the front edge of the sliding cabinet. (b) 0.501 m

55. (b) $60.0°$
57. (a) $M = (m/2)(2\mu_s \sin\theta - \cos\theta)(\cos\theta - \mu_s \sin\theta)^{-1}$
(b) $R = (m + M)g(1 + \mu_s^2)^{1/2}$,
$F = g[M^2 + \mu_s^2(m + M)^2]^{1/2}$
59. (a) 133 N (b) $n_A = 429$ N and $n_B = 257$ N
(c) $R_x = 133$ N and $R_y = -257$ N
61. 66.7 N
65. 1.09 m
67. (a) 4 500 N (b) 4.50×10^6 N/m^2 (c) yes.
69. (a) $P_y = (F_g/L)(d - ah/g)$ (b) 0.306 m
(c) $\mathbf{P} = (-306\mathbf{i} + 553\mathbf{j})$ N
71. $n_A = n_E = 6.66$ kN; $F_{AB} = 10.4$ kN $= F_{BC} = F_{DC} = F_{DE}$;
$F_{AC} = 7.94$ kN $= F_{CE}$; $F_{BD} = 15.9$ kN

Chapter 13

1. (a) 1.50 Hz, 0.667 s (b) 4.00 m (c) π rad (d) 2.83 m
3. (a) 20.0 cm (b) 94.2 cm/s as the particle passes through equilibrium (c) 17.8 m/s^2 at the maximum displacement from equilibrium
5. (b) 18.8 cm/s, 0.333 s (c) 178 cm/s^2, 0.500 s
(d) 12.0 cm
7. 0.627 s
9. (a) 40.0 cm/s, 160 cm/s^2 (b) 32.0 cm/s, -96.0 cm/s^2
(c) 0.232 s
11. 40.9 N/m
13. (a) 0.750 m (b) $x = -(0.750\text{ m})\sin(2.00t/\text{s})$
15. 0.628 m/s
17. 2.23 m/s
19. (a) 28.0 mJ (b) 1.02 m/s (c) 12.2 mJ (d) 15.8 mJ
21. (a) 2.61 m/s (b) 2.38 m/s
23. 2.60 cm and -2.60 cm
25. (a) 35.7 m (b) 29.1 s

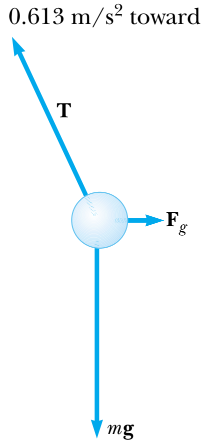


Chapter 13, Problem 57(a)

27. $\sim 10^0$ s
29. (a) 0.817 m/s (b) 2.54 rad/s^2 (c) 0.634 N
33. 0.944 kg·m^2
37. (a) 5.00×10^{-7} kg·m^2 (b) 3.16×10^{-4} N·m/rad
39. The x coordinate of the crank pin is $A\cos\omega t$.
41. 1.00×10^{-3} s^{-1}
43. (a) 2.95 Hz (b) 2.85 cm
47. Either 1.31 Hz or 0.641 Hz
49. 6.58 kN/m
51. (a) $2Mg$; $Mg(1 + y/L)$ (b) $T = (4\pi/3)(2L/g)^{1/2}$; 2.68 s
53. 6.62 cm
55. 9.19×10^{13} Hz
57. (a) See bottom of preceding column.
(b) $\dfrac{dT}{dt} = \dfrac{\pi(dM/dt)}{2\rho a^2 g^{1/2}[L_i + (dM/dt)t/2\rho a^2]^{1/2}}$
(c) $T = 2\pi g^{-1/2}[L_i + (dM/dt)t/2\rho a^2]^{1/2}$
59. $f = (2\pi L)^{-1}(gL + kh^2/M)^{1/2}$
61. (a) 3.56 Hz (b) 2.79 Hz (c) 2.10 Hz
63. (a) 3.00 s (b) 14.3 J (c) 25.5°
65. 0.224 rad/s

Chapter 14

1. $\sim 10^{-7}$ N toward you
3. $\mathbf{g} = (Gm/\ell^2)(\frac{1}{2} + \sqrt{2})$ toward the opposite corner
5. $(-100\mathbf{i} + 59.3\mathbf{j})$ pN
7. (a) 4.39×10^{20} N (b) 1.99×10^{20} N (c) 3.55×10^{22} N
9. 0.613 m/s^2 toward the Earth
11.

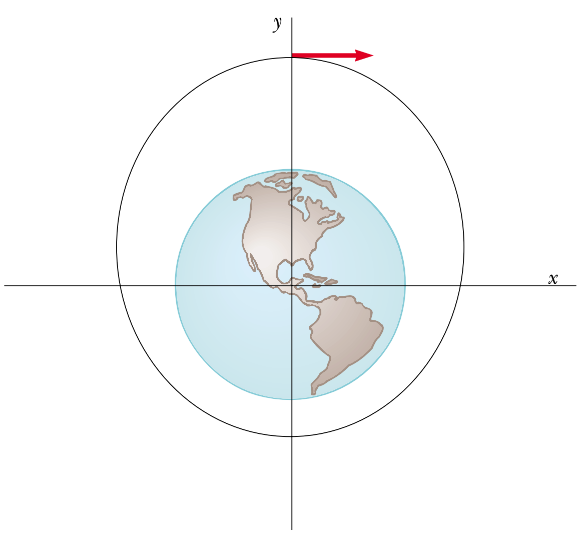


Either (1.000 m − 61.3 nm) or, if the objects have very high density, 247 mm.
15. 12.6×10^{31} kg
17. 1.27
19. 1.90×10^{27} kg
21. 8.92×10^7 m
25. $g = 2MGr(r^2 + a^2)^{-3/2}$ toward the center of mass
27. (a) -4.77×10^9 J (b) 569 N (c) 569 N up
29. (a) 1.84×10^9 kg/m^3 (b) 3.27×10^6 m/s^2
(c) -2.08×10^{13} J
31. (a) -1.67×10^{-14} J (b) At the center
33. 1.58×10^{10} J
35. (a) 1.48 h (b) 7.79 km/s (c) 6.43×10^9 J

37. 1.66×10^4 m/s
41. 15.6 km/s
43. $GM_E m/12R_E$
45. $2GmM/\pi R^2$ straight up in the picture
47. (a) 7.41×10^{-10} N (b) 1.04×10^{-8} N (c) 5.21×10^{-9} N
49. 2.26×10^{-7}
51. (b) 1.10×10^{32} kg
53. (b) $GMm/2R$
55. 7.79×10^{14} kg
57. 7.41×10^{-10} N
59. $v_{esc} = (8\pi G\rho/3)^{1/2} R$
61. (a) $v_1 = m_2(2G/d)^{1/2}(m_1 + m_2)^{-1/2}$
$v_2 = m_1(2G/d)^{1/2}(m_1 + m_2)^{-1/2}$
$v_{rel} = (2G/d)^{1/2}(m_1 + m_2)^{1/2}$
(b) $K_1 = 1.07 \times 10^{32}$ J, $K_2 = 2.67 \times 10^{31}$ J
63. (a) $A = M/\pi R^4$ (b) $F = GmM/r^2$ toward the center (c) $F = GmMr^2/R^4$ toward the center
65. 119 km
67. (a) -36.7 MJ (b) 9.24×10^{10} kg·m^2/s (c) 5.58 km/s, 10.4 Mm (d) 8.69 Mm (e) 134 min
71.

t (s)	x (m)	y (m)	v_x (m/s)	v_y (m/s)
0	0	12 740 000	5 000	0
10	50 000	12 740 000	4 999.9	− 24.6
20	99 999	12 739 754	4 999.7	− 49.1
30	149 996	12 739 263	4 999.4	− 73.7 . . .

The object does not hit the Earth; its minimum radius is $1.33R_E$. Its period is 1.09×10^4 s. A circular orbit would require speed 5.60 km/s.

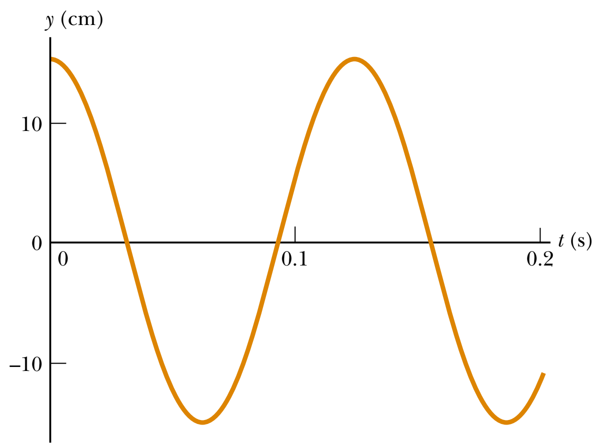


Chapter 15

1. 0.111 kg
3. 6.24 MPa
5. 5.27×10^{18} kg
7. 1.62 m
9. 7.74×10^{-3} m^2
11. 271 kN horizontally backward
13. $P_0 + (\rho d/2)(g^2 + a^2)^{1/2}$
15. 0.722 mm
17. 10.5 m; no, some alcohol and water evaporate.
19. 12.6 cm
21. 1.07 m^2
23. (a) 9.80 N (b) 6.17 N
25. (a) 7.00 cm (b) 2.80 kg
27. $\rho_{oil} = 1\,250$ kg/m^3; $\rho_{sphere} = 500$ kg/m^3
29. 1 430 m^3
31. 2.67×10^3 kg
33. (a) 1.06 m/s (b) 4.24 m/s
35. (a) 17.7 m/s (b) 1.73 mm
37. 31.6 m/s
39. 68.0 kPa
41. 103 m/s
43. (a) 4.43 m/s (b) The siphon can be no higher than 10.3 m.
45. $2\sqrt{h(h_0 - h)}$
47. 0.258 N
49. 1.91 m
53. 709 kg/m^3
55. top scale 17.3 N; bottom scale 31.7 N
59. 90.04%
61. 4.43 m/s
63. (a) 10.3 m (b) 0
65. (a) 18.3 mm (b) 14.3 mm (c) 8.56 mm
67. (a) 2.65 m/s (b) 2.31×10^4 Pa
69. (a) 1.25 cm (b) 13.8 m/s

Chapter 16

1. $y = 6\,[(x - 4.5t)^2 + 3]^{-1}$
3. (a) left (b) 5.00 m/s
5. (a) longitudinal (b) 665 s
7. (a) 156° (b) 0.058 4 cm
9. (a) y_1 in $+x$ direction, y_2 in $-x$ direction (b) 0.750 s (c) 1.00 m
11. 30.0 N
13. 1.64 m/s^2
15. 13.5 N
17. 586 m/s
19. 32.9 ms
21. 0.329 s
23. (a) See top of next page (b) 0.125 s
25. 0.319 m
27. 2.40 m/s
29. (a) 0.250 m (b) 40.0 rad/s (c) 0.300 rad/m (d) 20.9 m (e) 133 m/s (f) $+x$
31. (a) $y = (8.00\text{ cm})\sin(7.85x + 6\pi t)$
(b) $y = (8.00\text{ cm})\sin(7.85x + 6\pi t - 0.785)$
33. (a) 0.500 Hz, 3.14 rad/s (b) 3.14 rad/m
(c) $(0.100\text{ m})\sin(3.14x/\text{m} - 3.14t/\text{s})$



Chapter 16, Problem 23(a)

(d) $(0.100 \text{ m}) \sin(-3.14t/\text{s})$
(e) $(0.100 \text{ m}) \sin(4.71 \text{ rad} - 3.14t/\text{s})$ (f) 0.314 m/s

35. 2.00 cm, 2.98 m, 0.576 Hz, 1.72 m/s
37. (b) 3.18 Hz
41. 55.1 Hz
43. (a) 62.5 m/s (b) 7.85 m (c) 7.96 Hz (d) 21.1 W
45. (a) $A = 40.0$ (b) $A = 7.00$, $B = 0$, $C = 3.00$. One can take the dot product of the given equation with each one of **i**, **j**, and **k**. (c) By inspection, $A = 0$, $B = 7.00$ mm, $C = 3.00$/m, $D = 4.00$/s, $E = 2.00$. Consider the average value of both sides of the given equation to find A. Then consider the maximum value of both sides to find B. You can evaluate the partial derivative of both sides of the given equation with respect to x and separately with respect to t to obtain equations yielding C and D upon chosen substitutions for x and t. Then substitute $x = 0$ and $t = 0$ to obtain E.
47. It is if $v = (T/\mu)^{1/2}$
49. ~ 1 min
51. (a) $3.33\mathbf{i}$ m/s (b) -5.48 cm (c) 0.667 m, 5.00 Hz (d) 11.0 m/s
53. $(Lm/Mg \sin\theta)^{1/2}$
55. (a) 39.2 N (b) 0.892 m (c) 83.6 m/s
57. 14.7 kg
61. (a) $(0.707)2(L/g)^{1/2}$ (b) $L/4$
63. 3.86×10^{-4}
65. (a) $v = (2T_0/\mu_0)^{1/2} = v_0 2^{1/2}$
$v' = (2T_0/3\mu_0)^{1/2} = v_0 (2/3)^{1/2}$
(b) $0.966t_0$
67. 130 m/s, 1.73 km

Chapter 17

1. 5.56 km
3. 7.82 m
5. (a) 27.2 s (b) 25.7 s; the interval in (a) is longer
7. (a) 153 m/s (b) 614 m
9. (a) amplitude 2.00 μm, wavelength 40.0 cm, speed 54.6 m/s (b) -0.433 μm (c) 1.72 mm/s
11. $\Delta P = (0.2 \text{ Pa}) \sin(62.8x/\text{m} - 2.16 \times 10^4\, t/\text{s})$
13. (a) 6.52 mm (b) 20.5 m/s
15. 5.81 m
17. 66.0 dB
19. (a) 3.75 W/m^2 (b) 0.600 W/m^2
21. (a) 1.32×10^{-4} W/m^2 (b) 81.2 dB
23. 65.6 dB
25. (a) 65.0 dB (b) 67.8 dB (c) 69.6 dB
27. 1.13 μW
29. (a) 30.0 m (b) 9.49×10^5 m
31. (a) 332 J (b) 46.4 dB
33. (a) 75.7-Hz drop (b) 0.948 m
35. 26.4 m/s
37. 19.3 m
39. (a) 338 Hz (b) 483 Hz
41. 56.4°
43. (a) 56.3 s (b) 56.6 km farther along
45. 400 m; 27.5%
47. (a) 23.2 cm (b) 8.41×10^{-8} m (c) 1.38 cm
49. (a) 0.515/min (b) 0.614/min
51. 7.94 km
53. (a) 55.8 m/s (b) 2 500 Hz
55. Bat is gaining on the insect at the rate of 1.69 m/s.
57. (a)

(b) 0.343 m (c) 0.303 m (d) 0.383 m
(e) 1.03 kHz
59. (a) 0.691 m (b) 691 km
61. 1204.2 Hz
63. (a) 0.948° (b) 4.40°
65. 1.34×10^4 N
67. 95.5 s
69. (b) 531 Hz
71. (a) 6.45 (b) 0
73. $\sim 10^{11}$ Hz

Chapter 18

1. (a) 9.24 m (b) 600 Hz
3. 5.66 cm
5. 91.3°
7. (a) 2 (b) 9.28 m and 1.99 m
9. 15.7 m, 31.8 Hz, 500 m/s
11. At 0.089 1 m, 0.303 m, 0.518 m, 0.732 m, 0.947 m, and 1.16 m from one speaker
13. (a) 4.24 cm (b) 6.00 cm (c) 6.00 cm (d) 0.500 cm, 1.50 cm, and 2.50 cm
17. 0.786 Hz, 1.57 Hz, 2.36 Hz, and 3.14 Hz
19. (a) 163 N (b) 660 Hz
21. 19.976 kHz

23. 31.2 cm from the bridge; 3.84%
25. (a) 350 Hz (b) 400 kg
27. 0.352 Hz
29. (a) 3.66 m/s (b) 0.200 Hz
31. (a) 0.357 m (b) 0.715 m
33. (a) 531 Hz (b) 42.5 mm
35. around 3 kHz
37. n(206 Hz) for $n = 1$ to 9, and n(84.5 Hz) for $n = 2$ to 23
39. 239 s
41. 0.502 m and 0.837 m
43. (a) 350 m/s (b) 1.14 m
45. (a) 19.5 cm (b) 841 Hz
47. (a) 1.59 kHz (b) odd-numbered harmonics (c) 1.11 kHz
49. 5.64 beats/s
51. (a) 1.99 beats/s (b) 3.38 m/s
53. The second harmonic of E is close to the third harmonic of A, and the fourth harmonic of C# is close to the fifth harmonic of A.
55. (a) 3.33 rad (b) 283 Hz
57. 3.85 m/s away from the station or 3.77 m/s toward the station
59. 85.7 Hz
61. 31.1 N
63. (a) 59.9 Hz (b) 20.0 cm
65. (a) 1/2 (b) $[n/(n+1)]^2\, T$ (c) 9/16
67. 50.0 Hz, 1.70 m
69. (a) $2A\sin(2\pi x/\lambda)\cos(2\pi vt/\lambda)$
(b) $2A\sin(\pi x/L)\cos(\pi vt/L)$
(c) $2A\sin(2\pi x/L)\cos(2\pi vt/L)$
(d) $2A\sin(n\pi x/L)\cos(n\pi vt/L)$

Chapter 19

1. (a) 37.0°C = 310 K (b) −20.6°C = 253 K
3. (a) −274°C (b) 1.27 atm (c) 1.74 atm
5. (a) −320°F (b) 77.3 K
7. (a) 810°F (b) 450 K
9. 3.27 cm
11. (a) 3.005 8 m (b) 2.998 6 m
13. 55.0°C
15. (a) 0.109 cm^2 (b) increase
17. (a) 0.176 mm (b) 8.78 μm (c) 0.093 0 cm^3
19. (a) 2.52 MN/m^2 (b) It will not break.
21. 1.14°C
23. (a) 99.4 cm^3 (b) 0.943 cm
25. (a) 3.00 mol (b) 1.80×10^{24} molecules
27. 1.50×10^{29} molecules
29. 472 K
31. (a) 41.6 mol (b) 1.20 kg, in agreement with the tabulated density
33. (a) 400 kPa (b) 449 kPa
35. 2.27 kg
37. 3.67 cm^3
39. 4.39 kg
43. (a) 94.97 cm (b) 95.03 cm
45. 208°C
47. 3.55 cm
49. (a) Expansion makes density drop. (b) 5×10^{-5} (°C)$^{-1}$
51. (a) $h = nRT/(mg + P_0A)$ (b) 0.661 m
53. $\alpha\,\Delta T$ is much less than 1.
55. (a) 9.49×10^{-5} s (b) 57.4 s lost
57. (a) $\rho g P_0 V_i (P_0 + \rho g d)^{-1}$ (b) decrease (c) 10.3 m
61. (a) 5.00 MPa (b) 9.58×10^{-3}
63. 2.74 m
65. $L_c = 9.17$ cm, $L_s = 14.2$ cm
67. (a) $L_f = L_i e^{\alpha \Delta T}$ (b) 2.00×10^{-4}%; 59.4%
69. (a) 6.17×10^{-3} kg/m (b) 632 N (c) 580 N; 192 Hz

Chapter 20

1. (10.0 + 0.117)°C
3. 0.234 kJ/kg · °C
5. 29.6°C
7. (a) 0.435 cal/g · °C (b) beryllium
9. (a) 25.8°C (b) No
11. 50.7 ks
13. 0.294 g
15. 0.414 kg
17. (a) 0°C (b) 114 g
19. 59.4°C
21. 1.18 MJ
23. (a) $4P_iV_i$ (b) $T = (P_i/nRV_i)V^2$
25. 466 J
27. 810 J, 506 J, 203 J
29. $Q = -720$ J
31.

	Q	W	ΔE_{int}
BC	−	0	−
CA	−	−	−
AB	+	+	+

33. (a) 7.50 kJ (b) 900 K
35. 3.10 kJ; 37.6 kJ
37. (a) 0.041 0 m^3 (b) −5.48 kJ (c) −5.48 kJ
41. 2.40×10^6 cal/s
43. 10.0 kW
45. 51.2°C
47. (a) 0.89 ft^2 · °F · h/Btu (b) 1.85 ft^2 · °F · h/Btu (c) 2.08
49. (a) $\sim 10^3$ W (b) decreasing at $\sim 10^{-1}$ K/s
51. 364 K
53. 47.7 g
55. (a) 16.8 L (b) 0.351 L/s
57. 2.00 kJ/kg · °C
59. 1.87 kJ
61. (a) $4P_iV_i$ (b) $4P_iV_i$ (c) 9.08 kJ
63. 5.31 h
65. 872 g
67. (a) 15.0 mg. Block: $Q = 0$, $W = +5.00$ J, $\Delta E_{\text{int}} = 0$, $\Delta K = -5.00$ J; Ice: $Q = 0$, $W = -5.00$ J, $\Delta E_{\text{int}} = 5.00$ J, $\Delta K = 0$.

(b) 15.0 mg. Block: $Q = 0$, $W = 0$, $\Delta E_{\text{int}} = 5.00$ J, $\Delta K = -5.00$ J; Metal: $Q = 0$, $W = 0$, $\Delta E_{\text{int}} = 0$, $\Delta K = 0$. (c) 0.004 04°C. Moving slab: $Q = 0$, $W = +2.50$ J, $\Delta E_{\text{int}} = 2.50$ J, $\Delta K = -5.00$ J; Stationary slab: $Q = 0$, $W = -2.50$ J, $\Delta E_{\text{int}} = 2.50$ J, $\Delta K = 0$

69. 10.2 h

71. 9.32 kW

Chapter 21

1. 6.64×10^{-27} kg

3. 0.943 N; 1.57 Pa

5. 17.6 kPa

7. 3.32 mol

9. (a) 3.53×10^{23} atoms (b) 6.07×10^{-21} J (c) 1.35 km/s

11. (a) 8.76×10^{-21} J for both (b) 1.62 km/s for helium; 514 m/s for argon

13. 75.0 J

15. (a) 3.46 kJ (b) 2.45 kJ (c) 1.01 kJ

17. (a) 118 kJ (b) 6.03×10^3 kg

19. Between 10^{-2}°C and 10^{-3}°C

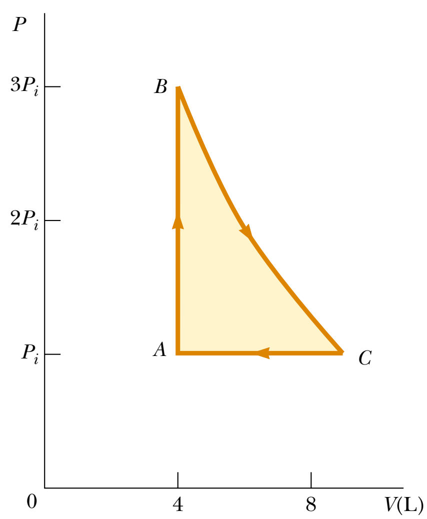
21. (a) 316 K (b) 200 J

23. $9\,P_iV_i$

25. (a) 1.39 atm (b) 366 K, 253 K (c) 0, 4.66 kJ, − 4.66 kJ

27. 227 K

29. (a)



(b) 8.79 L (c) 900 K (d) 300 K (e) 336 J

31. 25.0 kW

33. (a) 9.95 cal/K, 13.9 cal/K (b) 13.9 cal/K, 17.9 cal/K

35. 2.33×10^{-21} J

37. The ratio of oxygen to nitrogen molecules decreases to 85.5% of its sea-level value.

39. (a) 6.80 m/s (b) 7.41 m/s (c) 7.00 m/s

43. 819°C

45. (a) 3.21×10^{12} molecules (b) 778 km (c) $6.42 \times 10^{-4}\ \text{s}^{-1}$

49. (a) 9.36×10^{-8} m (b) 9.36×10^{-8} atm (c) 302 atm

51. (a) 100 kPa, 66.5 L, 400 K, 5.82 kJ, 7.48 kJ, 1.66 kJ (b) 133 kPa, 49.9 L, 400 K, 5.82 kJ, 5.82 kJ, 0 (c) 120 kPa, 41.6 L, 300 K, 0, − 910 J, − 910 J (d) 120 kPa, 43.3 L, 312 K, 722 J, 0, − 722 J

55. 0.625

57. (a) Pressure increases as volume decreases. (d) 0.500 atm^{-1}, 0.300 atm^{-1}

59. 1.09×10^{-3}; 2.69×10^{-2}; 0.529; 1.00; 0.199; 1.01×10^{-41}; 1.25×10^{-1082}

61. (a) Larger-mass molecules settle to the outside.

63. (a) 0.203 mol (b) $T_B = T_C = 900$ K; $V_C = 15.0$ L

(c, d)	P (atm)	V (L)	T (K)	E_{int} (kJ)
A	1	5	300	0.760
B	3	5	900	2.28
C	1	15	900	2.28

(e) For $A \rightarrow B$, lock the piston in place and put the cylinder into an oven at 900 K. For $B \rightarrow C$, keep the gas in the oven while gradually letting the gas expand to lift a load on the piston as far as it can. For $C \rightarrow A$, move the cylinder from the oven back to the 300-K room and let the gas cool and contract.

(f, g)	Q (kJ)	W (kJ)	ΔE_{int} (kJ)
$A \rightarrow B$	1.52	0	1.52
$B \rightarrow C$	1.67	1.67	0
$C \rightarrow A$	− 2.53	− 1.01	− 1.52
$ABCA$	0.656	0.656	0

65. (a) 3.34×10^{26} molecules (b) during the 27th day (c) 2.53×10^6

67. (a) 0.510 m/s (b) 20 ms

Chapter 22

1. (a) 6.94% (b) 335 J

3. (a) 10.7 kJ (b) 0.533 s

5. (a) 1.00 kJ (b) 0

7. (a) 67.2% (b) 58.8 kW

9. (a) 869 MJ (b) 330 MJ

11. (a) 741 J (b) 459 J

13. 0.330 or 33.0%

15. (a) 5.12% (b) 5.27 TJ/h (c) As conventional energy sources become more expensive, or as their true costs are recognized, alternative sources become economically viable.

17. (a) 214 J, 64.3 J
(b) − 35.7 J, − 35.7 J. The net effect is the transport of energy from the cold to the hot reservoir without expenditure of external work.
(c) 333 J, 233 J
(d) 83.3 J, 83.3 J, 0. The net effect is the expulsion of the energy entering the system by heat, entirely by work, in a cyclic process.
(e) − 0.111 J/K. The entropy of the Universe has decreased.

19. (a) 244 kPa (b) 192 J
21. 146 kW, 70.8 kW
23. 9.00
27. 72.2 J
29. (a) 24.0 J (b) 144 J
31. − 610 J/K
33. 195 J/K
35. 3.27 J/K
37. 1.02 kJ/K
39. 5.76 J/K. Temperature is constant if the gas is ideal.
41. 0.507 J/K
43. 18.4 J/K
45. (a) 1 (b) 6
47. (a)

Macrostate	Possible Microstates	Total Number of Microstates
All R	RRR	1
2R, 1G	RRG, RGR, GRR	3
1R, 2G	GRR, GRG, RGG	3
All G	GGG	1

(b)

Macrostate	Possible Microstates	Total Number of Microstates
All R	RRRRR	1
4R, 1G	RRRRG, RRRGR, RRGRR, RGRRR, GRRRR	5
3R, 2G	RRRGG, RRGRG, RGRRG, GRRRG, RRGGR, RGRGR, GRRGR, RGGRR, GRGRR, GGRRR	10
2R, 3G	GGGRR, GGRGR, GRGGR, RGGGR, GGRRG, GRGRG, RGGRG, GRRGG, RGRGG, RRGGG	10
1R, 4G	GGGGR, GGGRG, GGRGG, GRGGG, RGGGG	5
All G	GGGGG	1

49. 1.86
51. (a) 5.00 kW (b) 763 W
53. (a) $2nRT_i \ln 2$ (b) 0.273
55. 23.1 mW
57. 5.97×10^4 kg/s
59. (a) 3.19 cal/K (b) 98.19°F, 2.59 cal/K
61. 1.18 J/K
63. (a) $10.5nRT_i$ (b) $8.50nRT_i$ (c) 0.190 (d) 0.833
65. $nC_P \ln 3$
69. (a) 96.9 W = 8.33×10^4 cal/hr
(b) 1.19°C/h = 2.14°F/h

TABLE A.1 Conversion Factors

Length

	m	cm	km	in.	ft	mi
1 meter	1	10^{2}	10^{-3}	39.37	3.281	6.214×10^{-4}
1 centimeter	10^{-2}	1	10^{-5}	0.393 7	3.281×10^{-2}	6.214×10^{-6}
1 kilometer	10^{3}	10^{5}	1	3.937×10^{4}	3.281×10^{3}	0.621 4
1 inch	2.540×10^{-2}	2.540	2.540×10^{-5}	1	8.333×10^{-2}	1.578×10^{-5}
1 foot	0.304 8	30.48	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile	1 609	1.609×10^{5}	1.609	6.336×10^{4}	5 280	1

Mass

	kg	g	slug	u
1 kilogram	1	10^{3}	6.852×10^{-2}	6.024×10^{26}
1 gram	10^{-3}	1	6.852×10^{-5}	6.024×10^{23}
1 slug	14.59	1.459×10^{4}	1	8.789×10^{27}
1 atomic mass unit	1.660×10^{-27}	1.660×10^{-24}	1.137×10^{-28}	1

Note: 1 metric ton = 1 000 kg.

Time

	s	min	h	day	yr
1 second	1	1.667×10^{-2}	2.778×10^{-4}	1.157×10^{-5}	3.169×10^{-8}
1 minute	60	1	1.667×10^{-2}	6.994×10^{-4}	1.901×10^{-6}
1 hour	3 600	60	1	4.167×10^{-2}	1.141×10^{-4}
1 day	8.640×10^{4}	1 440	24	1	2.738×10^{-5}
1 year	3.156×10^{7}	5.259×10^{5}	8.766×10^{3}	365.2	1

Speed

	m/s	cm/s	ft/s	mi/h
1 meter per second	1	10^{2}	3.281	2.237
1 centimeter per second	10^{-2}	1	3.281×10^{-2}	2.237×10^{-2}
1 foot per second	0.304 8	30.48	1	0.681 8
1 mile per hour	0.447 0	44.70	1.467	1

Note: 1 mi/min = 60 mi/h = 88 ft/s.

continued

TABLE A.1 Continued

Force

	N	lb
1 newton	1	0.224 8
1 pound	4.448	1

Work, Energy, Heat

	J	ft·lb	eV
1 joule	1	0.737 6	6.242×10^{18}
1 ft·lb	1.356	1	8.464×10^{18}
1 eV	1.602×10^{-19}	1.182×10^{-19}	1
1 cal	4.186	3.087	2.613×10^{19}
1 Btu	1.055×10^{3}	7.779×10^{2}	6.585×10^{21}
1 kWh	3.600×10^{6}	2.655×10^{6}	2.247×10^{25}

	cal	Btu	kWh
1 joule	0.238 9	9.481×10^{-4}	2.778×10^{-7}
1 ft·lb	0.323 9	1.285×10^{-3}	3.766×10^{-7}
1 eV	3.827×10^{-20}	1.519×10^{-22}	4.450×10^{-26}
1 cal	1	3.968×10^{-3}	1.163×10^{-6}
1 Btu	2.520×10^{2}	1	2.930×10^{-4}
1 kWh	8.601×10^{5}	3.413×10^{2}	1

Pressure

	Pa	atm
1 pascal	1	9.869×10^{-6}
1 atmosphere	1.013×10^{5}	1
1 centimeter mercury[a]	1.333×10^{3}	1.316×10^{-2}
1 pound per inch^2	6.895×10^{3}	6.805×10^{-2}
1 pound per foot^2	47.88	4.725×10^{-4}

	cm Hg	lb/in.^{2}	lb/ft^2
1 newton per meter^2	7.501×10^{-4}	1.450×10^{-4}	2.089×10^{-2}
1 atmosphere	76	14.70	2.116×10^{3}
1 centimeter mercury[a]	1	0.194 3	27.85
1 pound per inch^2	5.171	1	144
1 pound per foot^2	3.591×10^{-2}	6.944×10^{-3}	1

[a] At 0°C and at a location where the acceleration due to gravity has its "standard" value, 9.806 65 m/s^2.

TABLE A.2 Symbols, Dimensions, and Units of Physical Quantities

Quantity	Common Symbol	Unit[a]	Dimensions[b]	Unit in Terms of Base SI Units
Acceleration	$\mathbf{a}$	m/s^2	L/T^2	m/s^2
Amount of substance	n	mole		mol
Angle	θ, ϕ	radian (rad)	1	
Angular acceleration	$\boldsymbol{\alpha}$	rad/s^2	T^{-2}	s^{-2}
Angular frequency	ω	rad/s	T^{-1}	s^{-1}
Angular momentum	$\mathbf{L}$	$kg \cdot m^2/s$	ML^2/T	$kg \cdot m^2/s$
Angular velocity	$\boldsymbol{\omega}$	rad/s	T^{-1}	s^{-1}
Area	A	m^2	L^2	m^2
Atomic number	Z			
Capacitance	C	farad (F)	Q^2T^2/ML^2	$A^2 \cdot s^4/kg \cdot m^2$
Charge	q, Q, e	coulomb (C)	Q	$A \cdot s$
Charge density				
Line	λ	C/m	Q/L	$A \cdot s/m$
Surface	σ	C/m^2	Q/L^2	$A \cdot s/m^2$
Volume	ρ	C/m^3	Q/L^3	$A \cdot s/m^3$
Conductivity	σ	$1/\Omega \cdot m$	Q^2T/ML^3	$A^2 \cdot s^3/kg \cdot m^3$
Current	I	AMPERE	Q/T	A
Current density	$\mathbf{J}$	A/m^2	Q/T^2	A/m^2
Density	ρ	kg/m^3	M/L^3	kg/m^3
Dielectric constant	κ			
Displacement	$\mathbf{r}, \mathbf{s}$	METER	L	m
Distance	d, h			
Length	ℓ, L			
Electric dipole moment	$\mathbf{p}$	$C \cdot m$	QL	$A \cdot s \cdot m$
Electric field	$\mathbf{E}$	V/m	ML/QT^2	$kg \cdot m/A \cdot s^3$
Electric flux	Φ_E	$V \cdot m$	ML^3/QT^2	$kg \cdot m^3/A \cdot s^3$
Electromotive force	$\mathcal{E}$	volt (V)	ML^2/QT^2	$kg \cdot m^2/A \cdot s^3$
Energy	E, U, K	joule (J)	ML^2/T^2	$kg \cdot m^2/s^2$
Entropy	S	J/K	$ML^2/T^2 \cdot K$	$kg \cdot m^2/s^2 \cdot K$
Force	$\mathbf{F}$	newton (N)	ML/T^2	$kg \cdot m/s^2$
Frequency	f	hertz (Hz)	T^{-1}	s^{-1}
Heat	Q	joule (J)	ML^2/T^2	$kg \cdot m^2/s^2$
Inductance	L	henry (H)	ML^2/Q^2	$kg \cdot m^2/A^2 \cdot s^2$
Magnetic dipole moment	$\boldsymbol{\mu}$	$N \cdot m/T$	QL^2/T	$A \cdot m^2$
Magnetic field	$\mathbf{B}$	tesla (T) $(= Wb/m^2)$	M/QT	$kg/A \cdot s^2$
Magnetic flux	Φ_B	weber (Wb)	ML^2/QT	$kg \cdot m^2/A \cdot s^2$
Mass	m, M	KILOGRAM	M	kg
Molar specific heat	C	$J/mol \cdot K$		$kg \cdot m^2/s^2 \cdot mol \cdot K$
Moment of inertia	I	$kg \cdot m^2$	ML^2	$kg \cdot m^2$
Momentum	$\mathbf{p}$	$kg \cdot m/s$	ML/T	$kg \cdot m/s$
Period	T	s	T	s
Permeability of space	μ_0	$N/A^2 (= H/m)$	ML/Q^2T	$kg \cdot m/A^2 \cdot s^2$
Permittivity of space	ϵ_0	$C^2/N \cdot m^2 (= F/m)$	Q^2T^2/ML^3	$A^2 \cdot s^4/kg \cdot m^3$
Potential	V	volt (V) $(= J/C)$	ML^2/QT^2	$kg \cdot m^2/A \cdot s^3$
Power	$\mathcal{P}$	watt (W) $(= J/s)$	ML^2/T^3	$kg \cdot m^2/s^3$

continued

TABLE A.2 Continued

Quantity	Common Symbol	Unit[a]	Dimensions[b]	Unit in Terms of Base SI Units
Pressure	P	pascal (Pa) = (N/m^2)	M/LT^2	kg/m·s^2
Resistance	R	ohm (Ω)(=V/A)	ML^2/Q^{2}T	kg·m^2/A^2·s^3
Specific heat	c	J/kg·K	L^2/T^2·K	m^2/s^2·K
Speed	v	m/s	L/T	m/s
Temperature	T	KELVIN	K	K
Time	t	SECOND	T	s
Torque	$\boldsymbol{\tau}$	N·m	ML^2/T^2	kg·m^2/s^2
Volume	V	m^3	L^3	m^3
Wavelength	λ	m	L	m
Work	W	joule (J)(=N·m)	ML^2/T^2	kg·m^2/s^2

[a] The base SI units are given in uppercase letters.

[b] The symbols M, L, T, and Q denote mass, length, time, and charge, respectively.

TABLE A.3 Table of Atomic Masses[a]

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
0	(Neutron)	n		1*	1.008 665		10.4 min
1	Hydrogen	H	1.007 9	1	1.007 825	99.985	
	Deuterium	D		2	2.014 102	0.015	
	Tritium	T		3*	3.016 049		12.33 yr
2	Helium	He	4.002 60	3	3.016 029	0.000 14	
				4	4.002 602	99.999 86	
				6*	6.018 886		0.81 s
3	Lithium	Li	6.941	6	6.015 121	7.5	
				7	7.016 003	92.5	
				8*	8.022 486		0.84 s
4	Beryllium	Be	9.012 2	7*	7.016 928		53.3 days
				9	9.012 174	100	
				10*	10.013 534		1.5×10^6 yr
5	Boron	B	10.81	10	10.012 936	19.9	
				11	11.009 305	80.1	
				12*	12.014 352		0.020 2 s
6	Carbon	C	12.011	10*	10.016 854		19.3 s
				11*	11.011 433		20.4 min
				12	12.000 000	98.90	
				13	13.003 355	1.10	
				14*	14.003 242		5 730 yr
				15*	15.010 599		2.45 s
7	Nitrogen	N	14.006 7	12*	12.018 613		0.011 0 s
				13*	13.005 738		9.96 min
				14	14.003 074	99.63	
				15	15.000 108	0.37	
				16*	16.006 100		7.13 s
				17*	17.008 450		4.17 s

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
8	Oxygen	O	15.999 4	14*	14.008 595		70.6 s
				15*	15.003 065		122 s
				16	15.994 915	99.761	
				17	16.999 132	0.039	
				18	17.999 160	0.20	
				19*	19.003 577		26.9 s
9	Fluorine	F	18.998 40	17*	17.002 094		64.5 s
				18*	18.000 937		109.8 min
				19	18.998 404	100	
				20*	19.999 982		11.0 s
				21*	20.999.950		4.2 s
10	Neon	Ne	20.180	18*	18.005 710		1.67 s
				19*	19.001 880		17.2 s
				20	19.992 435	90.48	
				21	20.993 841	0.27	
				22	21.991 383	9.25	
				23*	22.994 465		37.2 s
11	Sodium	Na	22.989 87	21*	20.997 650		22.5 s
				22*	21.994 434		2.61 yr
				23	22.989 770	100	
				24*	23.990 961		14.96 h
12	Magnesium	Mg	24.305	23*	22.994 124		11.3 s
				24	23.985 042	78.99	
				25	24.985 838	10.00	
				26	25.982 594	11.01	
				27*	26.984 341		9.46 min
13	Aluminum	Al	26.981 54	26*	25.986 892		7.4×10^5 yr
				27	26.981 538	100	
				28*	27.981 910		2.24 min
14	Silicon	Si	28.086	28	27.976 927	92.23	
				29	28.976 495	4.67	
				30	29.973 770	3.10	
				31*	30.975 362		2.62 h
				32*	31.974 148		172 yr
15	Phosphorus	P	30.973 76	30*	29.978 307		2.50 min
				31	30.973 762	100	
				32*	31.973 908		14.26 days
				33*	32.971 725		25.3 days
16	Sulfur	S	32.066	32	31.972 071	95.02	
				33	32.971 459	0.75	
				34	33.967 867	4.21	
				35*	34.969 033		87.5 days
				36	35.967 081	0.02	
17	Chlorine	Cl	35.453	35	34.968 853	75.77	
				36*	35.968 307		3.0×10^5 yr
				37	36.965 903	24.23	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
18	Argon	Ar	39.948	36	35.967 547	0.337	
				37*	36.966 776		35.04 days
				38	37.962 732	0.063	
				39*	38.964 314		269 yr
				40	39.962 384	99.600	
				42*	41.963 049		33 yr
19	Potassium	K	39.098 3	39	38.963 708	93.258 1	
				40*	39.964 000	0.011 7	1.28×10^9 yr
				41	40.961 827	6.730 2	
20	Calcium	Ca	40.08	40	39.962 591	96.941	
				41*	40.962 279		1.0×10^5 yr
				42	41.958 618	0.647	
				43	42.958 767	0.135	
				44	43.955 481	2.086	
				46	45.953 687	0.004	
				48	47.952 534	0.187	
21	Scandium	Sc	44.955 9	41*	40.969 250		0.596 s
				45	44.955 911	100	
22	Titanium	Ti	47.88	44*	43.959 691		49 yr
				46	45.952 630	8.0	
				47	46.951 765	7.3	
				48	47.947 947	73.8	
				49	48.947 871	5.5	
				50	49.944 792	5.4	
23	Vanadium	V	50.941 5	48*	47.952 255		15.97 days
				50*	49.947 161	0.25	1.5×10^{17} yr
				51	50.943 962	99.75	
24	Chromium	Cr	51.996	48*	47.954 033		21.6 h
				50	49.946 047	4.345	
				52	51.940 511	83.79	
				53	52.940 652	9.50	
				54	53.938 883	2.365	
25	Manganese	Mn	54.938 05	54*	53.940 361		312.1 days
				55	54.938 048	100	
26	Iron	Fe	55.847	54	53.939 613	5.9	
				55*	54.938 297		2.7 yr
				56	55.934 940	91.72	
				57	56.935 396	2.1	
				58	57.933 278	0.28	
				60*	59.934 078		1.5×10^6 yr
27	Cobalt	Co	58.933 20	59	58.933 198	100	
				60*	59.933 820		5.27 yr
28	Nickel	Ni	58.693	58	57.935 346	68.077	
				59*	58.934 350		7.5×10^4 yr
				60	59.930 789	26.223	
				61	60.931 058	1.140	
				62	61.928 346	3.634	
				63*	62.929 670		100 yr
				64	63.927 967	0.926	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
29	Copper	Cu	63.54	63	62.929 599	69.17	
				65	64.927 791	30.83	
30	Zinc	Zn	65.39	64	63.929 144	48.6	
				66	65.926 035	27.9	
				67	66.927 129	4.1	
				68	67.924 845	18.8	
				70	69.925 323	0.6	
31	Gallium	Ga	69.723	69	68.925 580	60.108	
				71	70.924 703	39.892	
32	Germanium	Ge	72.61	70	69.924 250	21.23	
				72	71.922 079	27.66	
				73	72.923 462	7.73	
				74	73.921 177	35.94	
				76	75.921 402	7.44	
33	Arsenic	As	74.921 6	75	74.921 594	100	
34	Selenium	Se	78.96	74	73.922 474	0.89	
				76	75.919 212	9.36	
				77	76.919 913	7.63	
				78	77.917 307	23.78	
				79*	78.918 497		$\leqslant 6.5 \times 10^4$ yr
				80	79.916 519	49.61	
				82*	81.916 697	8.73	1.4×10^{20} yr
35	Bromine	Br	79.904	79	78.918 336	50.69	
				81	80.916 287	49.31	
36	Krypton	Kr	83.80	78	77.920 400	0.35	
				80	79.916 377	2.25	
				81*	80.916 589		2.1×10^5 yr
				82	81.913 481	11.6	
				83	82.914 136	11.5	
				84	83.911 508	57.0	
				85*	84.912 531		10.76 yr
				86	85.910 615	17.3	
37	Rubidium	Rb	85.468	85	84.911 793	72.17	
				87*	86.909 186	27.83	4.75×10^{10} yr
38	Strontium	Sr	87.62	84	83.913 428	0.56	
				86	85.909 266	9.86	
				87	86.908 883	7.00	
				88	87.905 618	82.58	
				90*	89.907 737		29.1 yr
39	Yttrium	Y	88.905 8	89	88.905 847	100	
40	Zirconium	Zr	91.224	90	89.904 702	51.45	
				91	90.905 643	11.22	
				92	91.905 038	17.15	
				93*	92.906 473		1.5×10^6 yr
				94	93.906 314	17.38	
				96	95.908 274	2.80	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
41	Niobium	Nb	92.906 4	91*	90.906 988		6.8×10^2 yr
				92*	91.907 191		3.5×10^7 yr
				93	92.906 376	100	
				94*	93.907 280		2×10^4 yr
42	Molybdenum	Mo	95.94	92	91.906 807	14.84	
				93*	92.906 811		3.5×10^3 yr
				94	93.905 085	9.25	
				95	94.905 841	15.92	
				96	95.904 678	16.68	
				97	96.906 020	9.55	
				98	97.905 407	24.13	
				100	99.907 476	9.63	
43	Technetium	Tc		97*	96.906 363		2.6×10^6 yr
				98*	97.907 215		4.2×10^6 yr
				99*	98.906 254		2.1×10^5 yr
44	Ruthenium	Ru	101.07	96	95.907 597	5.54	
				98	97.905 287	1.86	
				99	98.905 939	12.7	
				100	99.904 219	12.6	
				101	100.905 558	17.1	
				102	101.904 348	31.6	
				104	103.905 428	18.6	
45	Rhodium	Rh	102.905 5	103	102.905 502	100	
46	Palladium	Pd	106.42	102	101.905 616	1.02	
				104	103.904 033	11.14	
				105	104.905 082	22.33	
				106	105.903 481	27.33	
				107*	106.905 126		6.5×10^6 yr
				108	107.903 893	26.46	
				110	109.905 158	11.72	
47	Silver	Ag	107.868	107	106.905 091	51.84	
				109	108.904 754	48.16	
48	Cadmium	Cd	112.41	106	105.906 457	1.25	
				108	107.904 183	0.89	
				109*	108.904 984		462 days
				110	109.903 004	12.49	
				111	110.904 182	12.80	
				112	111.902 760	24.13	
				113*	112.904 401	12.22	9.3×10^{15} yr
				114	113.903 359	28.73	
				116	115.904 755	7.49	
49	Indium	In	114.82	113	112.904 060	4.3	
				115*	114.903 876	95.7	4.4×10^{14} yr
50	Tin	Sn	118.71	112	111.904 822	0.97	
				114	113.902 780	0.65	
				115	114.903 345	0.36	
				116	115.901 743	14.53	
				117	116.902 953	7.68	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
(50)	(Tin)			118	117.901 605	24.22	
				119	118.903 308	8.58	
				120	119.902 197	32.59	
				121*	120.904 237		55 yr
				122	121.903 439	4.63	
				124	123.905 274	5.79	
51	Antimony	Sb	121.76	121	120.903 820	57.36	
				123	122.904 215	42.64	
				125*	124.905 251		2.7 yr
52	Tellurium	Te	127.60	120	119.904 040	0.095	
				122	121.903 052	2.59	
				123*	122.904 271	0.905	1.3×10^{13} yr
				124	123.902 817	4.79	
				125	124.904 429	7.12	
				126	125.903 309	18.93	
				128*	127.904 463	31.70	$>8 \times 10^{24}$ yr
				130*	129.906 228	33.87	$\leqslant 1.25 \times 10^{21}$ yr
53	Iodine	I	126.904 5	127	126.904 474	100	
				129*	128.904 984		1.6×10^{7} yr
54	Xenon	Xe	131.29	124	123.905 894	0.10	
				126	125.904 268	0.09	
				128	127.903 531	1.91	
				129	128.904 779	26.4	
				130	129.903 509	4.1	
				131	130.905 069	21.2	
				132	131.904 141	26.9	
				134	133.905 394	10.4	
				136*	135.907 215	8.9	$\geqslant 2.36 \times 10^{21}$ yr
55	Cesium	Cs	132.905 4	133	132.905 436	100	
				134*	133.906 703		2.1 yr
				135*	134.905 891		2×10^{6} yr
				137*	136.907 078		30 yr
56	Barium	Ba	137.33	130	129.906 289	0.106	
				132	131.905 048	0.101	
				133*	132.905 990		10.5 yr
				134	133.904 492	2.42	
				135	134.905 671	6.593	
				136	135.904 559	7.85	
				137	136.905 816	11.23	
				138	137.905 236	71.70	
57	Lanthanum	La	138.905	137*	136.906 462		6×10^{4} yr
				138*	137.907 105	0.090 2	1.05×10^{11} yr
				139	138.906 346	99.909 8	
58	Cerium	Ce	140.12	136	135.907 139	0.19	
				138	137.905 986	0.25	
				140	139.905 434	88.43	
				142*	141.909 241	11.13	$>5 \times 10^{16}$ yr
59	Praseodymium	Pr	140.907 6	141	140.907 647	100	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
60	Neodymium	Nd	144.24	142	141.907 718	27.13	
				143	142.909 809	12.18	
				144*	143.910 082	23.80	2.3×10^{15} yr
				145	144.912 568	8.30	
				146	145.913 113	17.19	
				148	147.916 888	5.76	
				150*	149.920 887	5.64	$>1 \times 10^{18}$ yr
61	Promethium	Pm		143*	142.910 928		265 days
				145*	144.912 745		17.7 yr
				146*	145.914 698		5.5 yr
				147*	146.915 134		2.623 yr
62	Samarium	Sm	150.36	144	143.911 996	3.1	
				146*	145.913 043		1.0×10^{8} yr
				147*	146.914 894	15.0	1.06×10^{11} yr
				148*	147.914 819	11.3	7×10^{15} yr
				149*	148.917 180	13.8	$>2 \times 10^{15}$ yr
				150	149.917 273	7.4	
				151*	150.919 928		90 yr
				152	151.919 728	26.7	
				154	153.922 206	22.7	
63	Europium	Eu	151.96	151	150.919 846	47.8	
				152*	151.921 740		13.5 yr
				153	152.921 226	52.2	
				154*	153.922 975		8.59 yr
				155*	154.922 888		4.7 yr
64	Gadolinium	Gd	157.25	148*	147.918 112		75 yr
				150*	149.918 657		1.8×10^{6} yr
				152*	151.919 787	0.20	1.1×10^{14} yr
				154	153.920 862	2.18	
				155	154.922 618	14.80	
				156	155.922 119	20.47	
				157	156.923 957	15.65	
				158	157.924 099	24.84	
				160	159.927 050	21.86	
65	Terbium	Tb	158.925 3	159	158.925 345	100	
66	Dysprosium	Dy	162.50	156	155.924 277	0.06	
				158	157.924 403	0.10	
				160	159.925 193	2.34	
				161	160.926 930	18.9	
				162	161.926 796	25.5	
				163	162.928 729	24.9	
				164	163.929 172	28.2	
67	Holmium	Ho	164.930 3	165	164.930 316	100	
				166*	165.932 282		1.2×10^{3} yr
68	Erbium	Er	167.26	162	161.928 775	0.14	
				164	163.929 198	1.61	
				166	165.930 292	33.6	

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
(68)	(Erbium)			167	166.932 047	22.95	
				168	167.932 369	27.8	
				170	169.935 462	14.9	
69	Thulium	Tm	168.934 2	169	168.934 213	100	
				171*	170.936 428		1.92 yr
70	Ytterbium	Yb	173.04	168	167.933 897	0.13	
				170	169.934 761	3.05	
				171	170.936 324	14.3	
				172	171.936 380	21.9	
				173	172.938 209	16.12	
				174	173.938 861	31.8	
				176	175.942 564	12.7	
71	Lutecium	Lu	174.967	173*	172.938 930		1.37 yr
				175	174.940 772	97.41	
				176*	175.942 679	2.59	3.78×10^{10} yr
72	Hafnium	Hf	178.49	174*	173.940 042	0.162	2.0×10^{15} yr
				176	175.941 404	5.206	
				177	176.943 218	18.606	
				178	177.943 697	27.297	
				179	178.945 813	13.629	
				180	179.946 547	35.100	
73	Tantalum	Ta	180.947 9	180	179.947 542	0.012	
				181	180.947 993	99.988	
74	Tungsten (Wolfram)	W	183.85	180	179.946 702	0.12	
				182	181.948 202	26.3	
				183	182.950 221	14.28	
				184	183.950 929	30.7	
				186	185.954 358	28.6	
75	Rhenium	Re	186.207	185	184.952 951	37.40	
				187*	186.955 746	62.60	4.4×10^{10} yr
76	Osmium	Os	190.2	184	183.952 486	0.02	
				186*	185.953 834	1.58	2.0×10^{15} yr
				187	186.955 744	1.6	
				188	187.955 832	13.3	
				189	188.958 139	16.1	
				190	189.958 439	26.4	
				192	191.961 468	41.0	
				194*	193.965 172		6.0 yr
77	Iridium	Ir	192.2	191	190.960 585	37.3	
				193	192.962 916	62.7	
78	Platinum	Pt	195.08	190*	189.959 926	0.01	6.5×10^{11} yr
				192	191.961 027	0.79	
				194	193.962 655	32.9	
				195	194.964 765	33.8	
				196	195.964 926	25.3	
				198	197.967 867	7.2	
79	Gold	Au	196.966 5	197	196.966 543	100	

continued

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
80	Mercury	Hg	200.59	196	195.965 806	0.15	
				198	197.966 743	9.97	
				199	198.968 253	16.87	
				200	199.968 299	23.10	
				201	200.970 276	13.10	
				202	201.970 617	29.86	
				204	203.973 466	6.87	
81	Thallium	Tl	204.383	203	202.972 320	29.524	
				204*	203.973 839		3.78 yr
				205	204.974 400	70.476	
		(Ra E″)		206*	205.976 084		4.2 min
		(Ac C″)		207*	206.977 403		4.77 min
		(Th C″)		208*	207.981 992		3.053 min
		(Ra C″)		210*	209.990 057		1.30 min
82	Lead	Pb	207.2	202*	201.972 134		5×10^4 yr
				204*	203.973 020	1.4	$\geqslant 1.4 \times 10^{17}$ yr
				205*	204.974 457		1.5×10^7 yr
				206	205.974 440	24.1	
				207	206.975 871	22.1	
				208	207.976 627	52.4	
		(Ra D)		210*	209.984 163		22.3 yr
		(Ac B)		211*	210.988 734		36.1 min
		(Th B)		212*	211.991 872		10.64 h
		(Ra B)		214*	213.999 798		26.8 min
83	Bismuth	Bi	208.980 3	207*	206.978 444		32.2 yr
				208*	207.979 717		3.7×10^5 yr
				209	208.980 374	100	
		(Ra E)		210*	209.984 096		5.01 days
		(Th C)		211*	210.987 254		2.14 min
				212*	211.991 259		60.6 min
		(Ra C)		214*	213.998 692		19.9 min
				215*	215.001 836		7.4 min
84	Polonium	Po		209*	208.982 405		102 yr
		(Ra F)		210*	209.982 848		138.38 days
		(Ac C′)		211*	210.986 627		0.52 s
		(Th C′)		212*	211.988 842		0.30 μs
		(Ra C′)		214*	213.995 177		164 μs
		(Ac A)		215*	214.999 418		0.001 8 s
		(Th A)		216*	216.001 889		0.145 s
		(Ra A)		218*	218.008 965		3.10 min
85	Astatine	At		215*	214.998 638		≈ 100 μs
				218*	218.008 685		1.6 s
				219*	219.011 294		0.9 min
86	Radon	Rn					
		(An)		219*	219.009 477		3.96 s
		(Tn)		220*	220.011 369		55.6 s
		(Rn)		222*	222.017 571		3.823 days
87	Francium	Fr					
		(Ac K)		223*	223.019 733		22 min

TABLE A.3 Continued

Atomic Number Z	Element	Symbol	Chemical Atomic Mass (u)	Mass Number (* Indicates Radioactive) A	Atomic Mass (u)	Percent Abundance	Half-Life (If Radioactive) $T_{1/2}$
88	Radium	Ra					
		(Ac X)		223*	223.018 499		11.43 days
		(Th X)		224*	224.020 187		3.66 days
		(Ra)		226*	226.025 402		1 600 yr
		(Ms Th_1)		228*	228.031 064		5.75 yr
89	Actinium	Ac		227*	227.027 749		21.77 yr
		(Ms Th_2)		228*	228.031 015		6.15 h
90	Thorium	Th	232.038 1				
		(Rd Ac)		227*	227.027 701		18.72 days
		(Rd Th)		228*	228.028 716		1.913 yr
				229*	229.031 757		7 300 yr
		(Io)		230*	230.033 127		75.000 yr
		(UY)		231*	231.036 299		25.52 h
		(Th)		232*	232.038 051	100	1.40×10^{10} yr
		(UX_1)		234*	234.043 593		24.1 days
91	Protactinium	Pa		231*	231.035 880		32.760 yr
		(Uz)		234*	234.043 300		6.7 h
92	Uranium	U	238.028 9	232*	232.037 131		69 yr
				233*	233.039 630		1.59×10^{5} yr
				234*	234.040 946	0.005 5	2.45×10^{5} yr
		(Ac U)		235*	235.043 924	0.720	7.04×10^{8} yr
				236*	236.045 562		2.34×10^{7} yr
		(UI)		238*	238.050 784	99.274 5	4.47×10^{9} yr
93	Neptunium	Np		235*	235.044 057		396 days
				236*	236.046 560		1.15×10^{5} yr
				237*	237.048 168		2.14×10^{6} yr
94	Plutonium	Pu		236*	236.046 033		2.87 yr
				238*	238.049 555		87.7 yr
				239*	239.052 157		2.412×10^{4} yr
				240*	240.053 808		6 560 yr
				241*	241.056 846		14.4 yr
				242*	242.058 737		3.73×10^{6} yr
				244*	244.064 200		8.1×10^{7} yr

[a] The masses in the sixth column are atomic masses, which include the mass of Z electrons. Data are from the National Nuclear Data Center, Brookhaven National Laboratory, prepared by Jagdish K. Tuli, July 1990. The data are based on experimental results reported in *Nuclear Data Sheets* and *Nuclear Physics* and also from *Chart of the Nuclides,* 14th ed. Atomic masses are based on those by A. H. Wapstra, G. Audi, and R. Hoekstra. Isotopic abundances are based on those by N. E. Holden.

APPENDIX B • Mathematics Review

These appendices in mathematics are intended as a brief review of operations and methods. Early in this course, you should be totally familiar with basic algebraic techniques, analytic geometry, and trigonometry. The appendices on differential and integral calculus are more detailed and are intended for those students who have difficulty applying calculus concepts to physical situations.

B.1 SCIENTIFIC NOTATION

Many quantities that scientists deal with often have very large or very small values. For example, the speed of light is about 300 000 000 m/s, and the ink required to make the dot over an *i* in this textbook has a mass of about 0.000 000 001 kg. Obviously, it is very cumbersome to read, write, and keep track of numbers such as these. We avoid this problem by using a method dealing with powers of the number 10:

$$10^0 = 1$$

$$10^1 = 10$$

$$10^2 = 10 \times 10 = 100$$

$$10^3 = 10 \times 10 \times 10 = 1000$$

$$10^4 = 10 \times 10 \times 10 \times 10 = 10\,000$$

$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 = 100\,000$$

and so on. The number of zeros corresponds to the power to which 10 is raised, called the **exponent** of 10. For example, the speed of light, 300 000 000 m/s, can be expressed as 3×10^8 m/s.

In this method, some representative numbers smaller than unity are

$$10^{-1} = \frac{1}{10} = 0.1$$

$$10^{-2} = \frac{1}{10 \times 10} = 0.01$$

$$10^{-3} = \frac{1}{10 \times 10 \times 10} = 0.001$$

$$10^{-4} = \frac{1}{10 \times 10 \times 10 \times 10} = 0.000\,1$$

$$10^{-5} = \frac{1}{10 \times 10 \times 10 \times 10 \times 10} = 0.000\,01$$

In these cases, the number of places the decimal point is to the left of the digit 1 equals the value of the (negative) exponent. Numbers expressed as some power of 10 multiplied by another number between 1 and 10 are said to be in **scientific notation.** For example, the scientific notation for 5 943 000 000 is 5.943×10^9 and that for 0.000 083 2 is 8.32×10^{-5}.

When numbers expressed in scientific notation are being multiplied, the following general rule is very useful:

$$10^n \times 10^m = 10^{n+m} \tag{B.1}$$

where n and m can be *any* numbers (not necessarily integers). For example, $10^2 \times 10^5 = 10^7$. The rule also applies if one of the exponents is negative: $10^3 \times 10^{-8} = 10^{-5}$.

When dividing numbers expressed in scientific notation, note that

$$\frac{10^n}{10^m} = 10^n \times 10^{-m} = 10^{n-m} \tag{B.2}$$

Exercises

With help from the above rules, verify the answers to the following:

1. $86\ 400 = 8.64 \times 10^4$
2. $9\ 816\ 762.5 = 9.816\ 762\ 5 \times 10^6$
3. $0.000\ 000\ 039\ 8 = 3.98 \times 10^{-8}$
4. $(4 \times 10^8)(9 \times 10^9) = 3.6 \times 10^{18}$
5. $(3 \times 10^7)(6 \times 10^{-12}) = 1.8 \times 10^{-4}$
6. $\dfrac{75 \times 10^{-11}}{5 \times 10^{-3}} = 1.5 \times 10^{-7}$
7. $\dfrac{(3 \times 10^6)(8 \times 10^{-2})}{(2 \times 10^{17})(6 \times 10^5)} = 2 \times 10^{-18}$

B.2 ALGEBRA

Some Basic Rules

When algebraic operations are performed, the laws of arithmetic apply. Symbols such as x, y, and z are usually used to represent quantities that are not specified, what are called the **unknowns.**

First, consider the equation

$$8x = 32$$

If we wish to solve for x, we can divide (or multiply) each side of the equation by the same factor without destroying the equality. In this case, if we divide both sides by 8, we have

$$\frac{8x}{8} = \frac{32}{8}$$

$$x = 4$$

Next consider the equation

$$x + 2 = 8$$

In this type of expression, we can add or subtract the same quantity from each side. If we subtract 2 from each side, we get

$$x + 2 - 2 = 8 - 2$$
$$x = 6$$

In general, if $x + a = b$, then $x = b - a$.

Now consider the equation

$$\frac{x}{5} = 9$$

If we multiply each side by 5, we are left with x on the left by itself and 45 on the right:

$$\left(\frac{x}{5}\right)(5) = 9 \times 5$$
$$x = 45$$

In all cases, *whatever operation is performed on the left side of the equality must also be performed on the right side.*

The following rules for multiplying, dividing, adding, and subtracting fractions should be recalled, where a, b, and c are three numbers:

	Rule	Example
Multiplying	$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$	$\left(\frac{2}{3}\right)\left(\frac{4}{5}\right) = \frac{8}{15}$
Dividing	$\frac{(a/b)}{(c/d)} = \frac{ad}{bc}$	$\frac{2/3}{4/5} = \frac{(2)(5)}{(4)(3)} = \frac{10}{12}$
Adding	$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$	$\frac{2}{3} - \frac{4}{5} = \frac{(2)(5) - (4)(3)}{(3)(5)} = -\frac{2}{15}$

Exercises

In the following exercises, solve for x:

	Answers
1. $a = \frac{1}{1 + x}$	$x = \frac{1 - a}{a}$
2. $3x - 5 = 13$	$x = 6$
3. $ax - 5 = bx + 2$	$x = \frac{7}{a - b}$
4. $\frac{5}{2x + 6} = \frac{3}{4x + 8}$	$x = -\frac{11}{7}$

Powers

When powers of a given quantity x are multiplied, the following rule applies:

$$x^n x^m = x^{n+m} \qquad \textbf{(B.3)}$$

For example, $x^2x^4 = x^{2+4} = x^6$.

When dividing the powers of a given quantity, the rule is

$$\frac{x^n}{x^m} = x^{n-m} \tag{B.4}$$

For example, $x^8/x^2 = x^{8-2} = x^6$.

A power that is a fraction, such as $\frac{1}{3}$, corresponds to a root as follows:

$$x^{1/n} = \sqrt[n]{x} \tag{B.5}$$

For example, $4^{1/3} = \sqrt[3]{4} = 1.5874$. (A scientific calculator is useful for such calculations.)

Finally, any quantity x^n raised to the mth power is

$$(x^n)^m = x^{nm} \tag{B.6}$$

Table B.1 summarizes the rules of exponents.

TABLE B.1
Rules of Exponents

$x^0 = 1$
$x^1 = x$
$x^n x^m = x^{n+m}$
$x^n/x^m = x^{n-m}$
$x^{1/n} = \sqrt[n]{x}$
$(x^n)^m = x^{nm}$

Exercises

Verify the following:

1. $3^2 \times 3^3 = 243$
2. $x^5x^{-8} = x^{-3}$
3. $x^{10}/x^{-5} = x^{15}$
4. $5^{1/3} = 1.709\,975$ (Use your calculator.)
5. $60^{1/4} = 2.783\,158$ (Use your calculator.)
6. $(x^4)^3 = x^{12}$

Factoring

Some useful formulas for factoring an equation are

$$ax + ay + az = a(x + y + x) \qquad \text{common factor}$$

$$a^2 + 2ab + b^2 = (a + b)^2 \qquad \text{perfect square}$$

$$a^2 - b^2 = (a + b)(a - b) \qquad \text{differences of squares}$$

Quadratic Equations

The general form of a quadratic equation is

$$ax^2 + bx + c = 0 \tag{B.7}$$

where x is the unknown quantity and a, b, and c are numerical factors referred to as **coefficients** of the equation. This equation has two roots, given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{B.8}$$

If $b^2 \geq 4ac$, the roots are real.

EXAMPLE 1

The equation $x^2 + 5x + 4 = 0$ has the following roots corresponding to the two signs of the square-root term:

$$x = \frac{-5 \pm \sqrt{5^2 - (4)(1)(4)}}{2(1)} = \frac{-5 \pm \sqrt{9}}{2} = \frac{-5 \pm 3}{2}$$

$$x_+ = \frac{-5 + 3}{2} = \boxed{-1} \qquad x_- = \frac{-5 - 3}{2} = \boxed{-4}$$

where x_+ refers to the root corresponding to the positive sign and x_- refers to the root corresponding to the negative sign.

EXERCISES

Solve the following quadratic equations:

	Answers	
1. $x^2 + 2x - 3 = 0$	$x_+ = 1$	$x_- = -3$
2. $2x^2 - 5x + 2 = 0$	$x_+ = 2$	$x_- = \frac{1}{2}$
3. $2x^2 - 4x - 9 = 0$	$x_+ = 1 + \sqrt{22}/2$	$x_- = 1 - \sqrt{22}/2$

Linear Equations

A linear equation has the general form

$$y = mx + b \tag{B.9}$$

where m and b are constants. This equation is referred to as being linear because the graph of y versus x is a straight line, as shown in Figure B.1. The constant b, called the ***y*-intercept,** represents the value of y at which the straight line intersects the y axis. The constant m is equal to the **slope** of the straight line and is also equal to the tangent of the angle that the line makes with the x axis. If any two points on the straight line are specified by the coordinates (x_1, y_1) and (x_2, y_2), as in Figure B.1, then the slope of the straight line can be expressed as

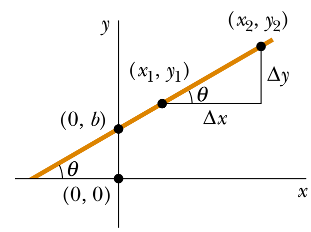


Figure B.1

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \tan\theta \tag{B.10}$$

Note that m and b can have either positive or negative values. If $m > 0$, the straight line has a *positive* slope, as in Figure B1. If $m < 0$, the straight line has a *negative* slope. In Figure B.1, both m and b are positive. Three other possible situations are shown in Figure B.2.

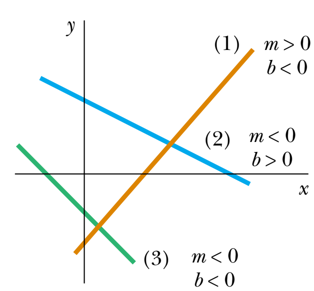


Figure B.2

EXERCISES

1. Draw graphs of the following straight lines:
   (a) $y = 5x + 3$ (b) $y = -2x + 4$ (c) $y = -3x - 6$
2. Find the slopes of the straight lines described in Exercise 1.

Answers (a) 5 (b) -2 (c) -3

3. Find the slopes of the straight lines that pass through the following sets of points:
(a) $(0, -4)$ and $(4, 2)$, (b) $(0, 0)$ and $(2, -5)$, and (c) $(-5, 2)$ and $(4, -2)$

Answers (a) $3/2$ (b) $-5/2$ (c) $-4/9$

Solving Simultaneous Linear Equations

Consider the equation $3x + 5y = 15$, which has two unknowns, x and y. Such an equation does not have a unique solution. For example, note that $(x = 0, y = 3)$, $(x = 5, y = 0)$, and $(x = 2, y = 9/5)$ are all solutions to this equation.

If a problem has two unknowns, a unique solution is possible only if we have *two* equations. In general, if a problem has n unknowns, its solution requires n equations. In order to solve two simultaneous equations involving two unknowns, x and y, we solve one of the equations for x in terms of y and substitute this expression into the other equation.

EXAMPLE 2

Solve the following two simultaneous equations:

$$(1) \quad 5x + y = -8$$

$$(2) \quad 2x - 2y = 4$$

Solution From (2), $x = y + 2$. Substitution of this into (1) gives

$$5(y + 2) + y = -8$$

$$6y = -18$$

$$y = -3$$

$$x = y + 2 = \boxed{-1}$$

Alternate Solution Multiply each term in (1) by the factor 2 and add the result to (2):

$$10x + 2y = -16$$

$$\underline{2x - 2y = 4}$$

$$12x = -12$$

$$x = -1$$

$$y = x - 2 = \boxed{-3}$$

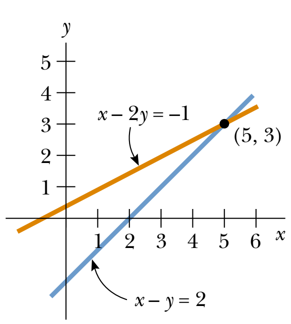


Figure B.3

Two linear equations containing two unknowns can also be solved by a graphical method. If the straight lines corresponding to the two equations are plotted in a conventional coordinate system, the intersection of the two lines represents the solution. For example, consider the two equations

$$x - y = 2$$

$$x - 2y = -1$$

These are plotted in Figure B.3. The intersection of the two lines has the coordinates $x = 5$, $y = 3$. This represents the solution to the equations. You should check this solution by the analytical technique discussed above.

EXERCISES

Solve the following pairs of simultaneous equations involving two unknowns:

Answers

1. $x + y = 8$ $\quad\quad$ $x = 5, y = 3$
   $x - y = 2$

2. $98 - T = 10a$ $\qquad T = 65,\ a = 3.27$
   $T - 49 = 5a$
3. $6x + 2y = 6$ $\qquad x = 2,\ y = -3$
   $8x - 4y = 28$

Logarithms

Suppose that a quantity x is expressed as a power of some quantity a:

$$x = a^y \tag{B.11}$$

The number a is called the **base** number. The **logarithm** of x with respect to the base a is equal to the exponent to which the base must be raised in order to satisfy the expression $x = a^y$:

$$y = \log_a x \tag{B.12}$$

Conversely, the **antilogarithm** of y is the number x:

$$x = \text{antilog}_a\, y \tag{B.13}$$

In practice, the two bases most often used are base 10, called the *common* logarithm base, and base $e = 2.718 \ldots$, called Euler's constant or the *natural* logarithm base. When common logarithms are used,

$$y = \log_{10} x \qquad (\text{or } x = 10^y) \tag{B.14}$$

When natural logarithms are used,

$$y = \ln_e x \qquad (\text{or } x = e^y) \tag{B.15}$$

For example, $\log_{10} 52 = 1.716$, so that $\text{antilog}_{10}\, 1.716 = 10^{1.716} = 52$. Likewise, $\ln_e 52 = 3.951$, so $\text{antiln}_e\, 3.951 = e^{3.951} = 52$.

In general, note that you can convert between base 10 and base e with the equality

$$\ln_e x = (2.302\,585) \log_{10} x \tag{B.16}$$

Finally, some useful properties of logarithms are

$$\begin{aligned}
\log(ab) &= \log a + \log b \\
\log(a/b) &= \log a - \log b \\
\log(a^n) &= n \log a \\
\ln e &= 1 \\
\ln e^a &= a \\
\ln\left(\frac{1}{a}\right) &= -\ln a
\end{aligned}$$

B.3 GEOMETRY

The **distance** d between two points having coordinates (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{B.17}$$

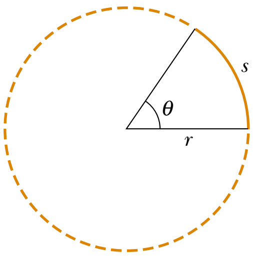


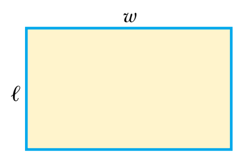
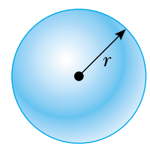
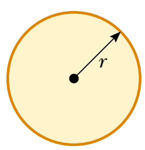
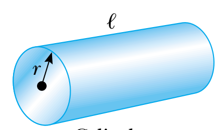
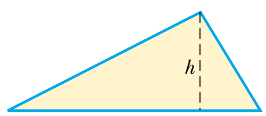
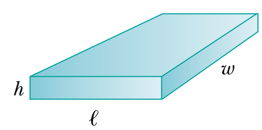
Figure B.4

Radian measure: The arc length s of a circular arc (Fig. B.4) is proportional to the radius r for a fixed value of θ (in radians):

$$s = r\theta$$
$$\theta = \frac{s}{r} \tag{B.18}$$

Table B.2 gives the areas and volumes for several geometric shapes used throughout this text:

TABLE B.2 Useful Information for Geometry

Shape	Area or Volume	Shape	Area or Volume
 Rectangle	Area = ℓw	 Sphere	Surface area = $4\pi r^2$ Volume = $\frac{4\pi r^3}{3}$
 Circle	Area = πr^2 (Circumference = $2\pi r$)	 Cylinder	Lateral surface area = $2\pi r\ell$ Volume = $\pi r^2\ell$
 Triangle	Area = $\frac{1}{2}bh$	 Rectangular box	Area = $2(\ell h + \ell w + hw)$ Volume = ℓwh

The equation of a **straight line** (Fig. B.5) is

$$y = mx + b \tag{B.19}$$

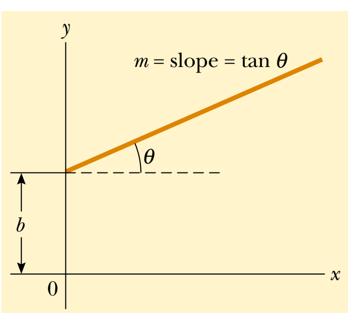


Figure B.5

where b is the y-intercept and m is the slope of the line.

The equation of a **circle** of radius R centered at the origin is

$$x^2 + y^2 = R^2 \tag{B.20}$$

The equation of an **ellipse** having the origin at its center (Fig. B.6) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{B.21}$$

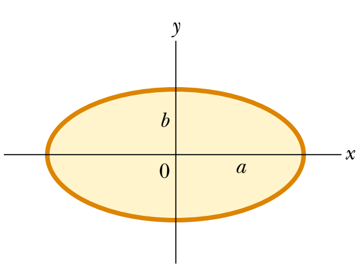


Figure B.6

where a is the length of the semi-major axis (the longer one) and b is the length of the semi-minor axis (the shorter one).

The equation of a **parabola** the vertex of which is at $y = b$ (Fig. B.7) is

$$y = ax^2 + b \tag{B.22}$$

The equation of a **rectangular hyperbola** (Fig. B.8) is

$$xy = \text{constant} \tag{B.23}$$

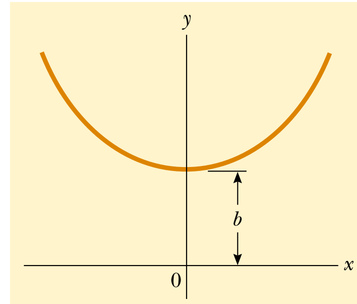


Figure B.7

B.4 TRIGONOMETRY

That portion of mathematics based on the special properties of the right triangle is called trigonometry. By definition, a right triangle is one containing a 90° angle. Consider the right triangle shown in Figure B.9, where side a is opposite the angle θ, side b is adjacent to the angle θ, and side c is the hypotenuse of the triangle. The three basic trigonometric functions defined by such a triangle are the sine (sin), cosine (cos), and tangent (tan) functions. In terms of the angle θ, these functions are defined by

$$\sin\theta \equiv \frac{\text{side opposite }\theta}{\text{hypotenuse}} = \frac{a}{c} \tag{B.24}$$

$$\cos\theta \equiv \frac{\text{side adjacent to }\theta}{\text{hypotenuse}} = \frac{b}{c} \tag{B.25}$$

$$\tan\theta \equiv \frac{\text{side opposite }\theta}{\text{side adjacent to }\theta} = \frac{a}{b} \tag{B.26}$$

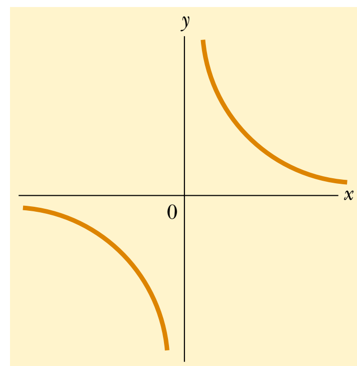


Figure B.8

The Pythagorean theorem provides the following relationship between the sides of a right triangle:

$$c^2 = a^2 + b^2 \tag{B.27}$$

From the above definitions and the Pythagorean theorem, it follows that

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

The cosecant, secant, and cotangent functions are defined by

$$\csc\theta \equiv \frac{1}{\sin\theta} \qquad \sec\theta \equiv \frac{1}{\cos\theta} \qquad \cot\theta \equiv \frac{1}{\tan\theta}$$

The relationships below follow directly from the right triangle shown in Figure B.9:

$$\sin\theta = \cos(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta)$$

$$\cot\theta = \tan(90° - \theta)$$

Some properties of trigonometric functions are

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

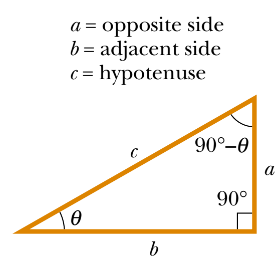


Figure B.9

The following relationships apply to *any* triangle, as shown in Figure B.10:

$$\alpha + \beta + \gamma = 180°$$

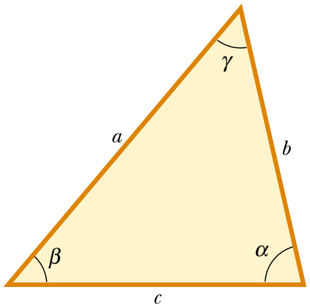


Figure B.10

Law of cosines
$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Law of sines
$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

Table B.3 lists a number of useful trigonometric identities.

TABLE B.3 Some Trigonometric Identities

$\sin^2\theta + \cos^2\theta = 1$	$\csc^2\theta = 1 + \cot^2\theta$
$\sec^2\theta = 1 + \tan^2\theta$	$\sin^2\dfrac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$
$\sin 2\theta = 2\sin\theta\cos\theta$	$\cos^2\dfrac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)$
$\cos 2\theta = \cos^2\theta - \sin^2\theta$	$1 - \cos\theta = 2\sin^2\dfrac{\theta}{2}$
$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$	$\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1 - \cos\theta}{1 + \cos\theta}}$
$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$	
$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$	

EXAMPLE 3

Consider the right triangle in Figure B.11, in which $a = 2$, $b = 5$, and c is unknown. From the Pythagorean theorem, we have

$$c^2 = a^2 + b^2 = 2^2 + 5^2 = 4 + 25 = 29$$

$$c = \sqrt{29} = \boxed{5.39}$$

To find the angle θ, note that

$$\tan\theta = \frac{a}{b} = \frac{2}{5} = 0.400$$

From a table of functions or from a calculator, we have

$$\theta = \tan^{-1}(0.400) = \boxed{21.8°}$$

where $\tan^{-1}(0.400)$ is the notation for "angle whose tangent is 0.400," sometimes written as arctan (0.400).

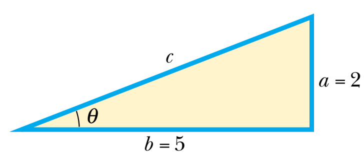


Figure B.11

EXERCISES

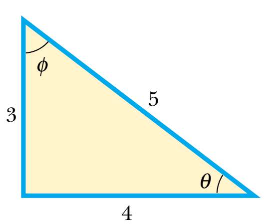


Figure B.12

1. In Figure B.12, identify (a) the side opposite θ and (b) the side adjacent to ϕ and then find (c) $\cos\theta$, (d) $\sin\phi$, and (e) $\tan\phi$.

Answers (a) 3, (b) 3, (c) $\frac{4}{5}$, (d) $\frac{4}{5}$, and (e) $\frac{4}{3}$

2. In a certain right triangle, the two sides that are perpendicular to each other are 5 m and 7 m long. What is the length of the third side?

Answer 8.60 m

3. A right triangle has a hypotenuse of length 3 m, and one of its angles is 30°. What is the length of (a) the side opposite the 30° angle and (b) the side adjacent to the 30° angle?

Answers (a) 1.5 m, (b) 2.60 m

B.5 SERIES EXPANSIONS

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\ln(1 \pm x) = \pm x - \tfrac{1}{2}x^2 \pm \tfrac{1}{3}x^3 - \cdots$$

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \\
\tan x &= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \qquad |x| < \pi/2
\end{aligned}\right\} x \text{ in radians}$$

For $x \ll 1$, the following approximations can be used[1]:

$$(1+x)^n \approx 1 + nx \qquad \sin x \approx x$$

$$e^x \approx 1 + x \qquad \cos x \approx 1$$

$$\ln(1 \pm x) \approx \pm x \qquad \tan x \approx x$$

B.6 DIFFERENTIAL CALCULUS

In various branches of science, it is sometimes necessary to use the basic tools of calculus, invented by Newton, to describe physical phenomena. The use of calculus is fundamental in the treatment of various problems in Newtonian mechanics, electricity, and magnetism. In this section, we simply state some basic properties and "rules of thumb" that should be a useful review to the student.

First, a **function** must be specified that relates one variable to another (such as a coordinate as a function of time). Suppose one of the variables is called y (the dependent variable), the other x (the independent variable). We might have a function relationship such as

$$y(x) = ax^3 + bx^2 + cx + d$$

If a, b, c, and d are specified constants, then y can be calculated for any value of x. We usually deal with continuous functions, that is, those for which y varies "smoothly" with x.

[1] The approximations for the functions $\sin x$, $\cos x$, and $\tan x$ are for $x \leq 0.1$ rad.

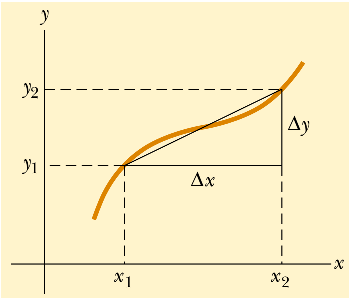


Figure B.13

The **derivative** of y with respect to x is defined as the limit, as Δx approaches zero, of the slopes of chords drawn between two points on the y versus x curve. Mathematically, we write this definition as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x} \qquad \textbf{(B.28)}$$

where Δy and Δx are defined as $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ (Fig. B.13). It is important to note that dy/dx *does not* mean dy divided by dx, but is simply a notation of the limiting process of the derivative as defined by Equation B.28.

A useful expression to remember when $y(x) = ax^n$, where a is a *constant* and n is *any* positive or negative number (integer or fraction), is

$$\frac{dy}{dx} = nax^{n-1} \qquad \textbf{(B.29)}$$

If $y(x)$ is a polynomial or algebraic function of x, we apply Equation B.29 to *each* term in the polynomial and take $d[\text{constant}]/dx = 0$. In Examples 4 through 7, we evaluate the derivatives of several functions.

EXAMPLE 4

Suppose $y(x)$ (that is, y as a function of x) is given by

$$y(x) = ax^3 + bx + c$$

where a and b are constants. Then it follows that

$$y(x + \Delta x) = a(x + \Delta x)^3 + b(x + \Delta x) + c$$

$$y(x + \Delta x) = a(x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) + b(x + \Delta x) + c$$

so

$$\Delta y = y(x + \Delta x) - y(x) = a(3x^2\Delta x + 3x\Delta x^2 + \Delta x^3) + b\Delta x$$

Substituting this into Equation B.28 gives

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [3ax^2 + 3x\Delta x + \Delta x^2] + b$$

$$\frac{dy}{dx} = \boxed{3ax^2 + b}$$

EXAMPLE 5

$$y(x) = 8x^5 + 4x^3 + 2x + 7$$

Solution Applying Equation B.29 to each term independently, and remembering that d/dx (constant) $= 0$, we have

$$\frac{dy}{dx} = 8(5)x^4 + 4(3)x^2 + 2(1)x^0 + 0$$

$$\frac{dy}{dx} = \boxed{40x^4 + 12x^2 + 2}$$

Special Properties of the Derivative

A. Derivative of the product of two functions If a function $f(x)$ is given by the product of two functions, say, $g(x)$ and $h(x)$, then the derivative of $f(x)$ is defined as

$$\frac{d}{dx} f(x) = \frac{d}{dx}[g(x)h(x)] = g\frac{dh}{dx} + h\frac{dg}{dx} \qquad \textbf{(B.30)}$$

B. Derivative of the sum of two functions If a function $f(x)$ is equal to the sum of two functions, then the derivative of the sum is equal to the sum of the derivatives:

$$\frac{d}{dx} f(x) = \frac{d}{dx}[g(x) + h(x)] = \frac{dg}{dx} + \frac{dh}{dx} \qquad \textbf{(B.31)}$$

C. Chain rule of differential calculus If $y = f(x)$ and $x = g(z)$, then dy/dz can be written as the product of two derivatives:

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} \qquad \textbf{(B.32)}$$

D. The second derivative The second derivative of y with respect to x is defined as the derivative of the function dy/dx (the derivative of the derivative). It is usually written

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \qquad \textbf{(B.33)}$$

EXAMPLE 6

Find the derivative of $y(x) = x^3/(x+1)^2$ with respect to x.

Solution We can rewrite this function as $y(x) = x^3(x+1)^{-2}$ and apply Equation B.30:

$$\frac{dy}{dx} = (x+1)^{-2}\frac{d}{dx}(x^3) + x^3\frac{d}{dx}(x+1)^{-2}$$

$$= (x+1)^{-2}3x^2 + x^3(-2)(x+1)^{-3}$$

$$\frac{dy}{dx} = \frac{3x^2}{(x+1)^2} - \frac{2x^3}{(x+1)^3}$$

EXAMPLE 7

A useful formula that follows from Equation B.30 is the derivative of the quotient of two functions. Show that

$$\frac{d}{dx}\left[\frac{g(x)}{h(x)}\right] = \frac{h\dfrac{dg}{dx} - g\dfrac{dh}{dx}}{h^2}$$

Solution We can write the quotient as gh^{-1} and then apply Equations B.29 and B.30:

$$\frac{d}{dx}\left(\frac{g}{h}\right) = \frac{d}{dx}(gh^{-1}) = g\frac{d}{dx}(h^{-1}) + h^{-1}\frac{d}{dx}(g)$$

$$= -gh^{-2}\frac{dh}{dx} + h^{-1}\frac{dg}{dx}$$

$$= \frac{h\dfrac{dg}{dx} - g\dfrac{dh}{dx}}{h^2}$$

Some of the more commonly used derivatives of functions are listed in Table B.4.

B.7 INTEGRAL CALCULUS

We think of integration as the inverse of differentiation. As an example, consider the expression

$$f(x) = \frac{dy}{dx} = 3ax^2 + b \qquad \textbf{(B.34)}$$

which was the result of differentiating the function

$$y(x) = ax^3 + bx + c$$

TABLE B.4
Derivatives for Several Functions

$$\frac{d}{dx}(a) = 0$$
$$\frac{d}{dx}(ax^n) = nax^{n-1}$$
$$\frac{d}{dx}(e^{ax}) = ae^{ax}$$
$$\frac{d}{dx}(\sin ax) = a\cos ax$$
$$\frac{d}{dx}(\cos ax) = -a\sin ax$$
$$\frac{d}{dx}(\tan ax) = a\sec^2 ax$$
$$\frac{d}{dx}(\cot ax) = -a\csc^2 ax$$
$$\frac{d}{dx}(\sec x) = \tan x \sec x$$
$$\frac{d}{dx}(\csc x) = -\cot x \csc x$$
$$\frac{d}{dx}(\ln ax) = \frac{1}{x}$$

Note: The letters a and n are constants.

in Example 4. We can write Equation B.34 as $dy = f(x)\,dx = (3ax^2 + b)\,dx$ and obtain $y(x)$ by "summing" over all values of x. Mathematically, we write this inverse operation

$$y(x) = \int f(x)\,dx$$

For the function $f(x)$ given by Equation B.34, we have

$$y(x) = \int (3ax^2 + b)\,dx = ax^3 + bx + c$$

where c is a constant of the integration. This type of integral is called an *indefinite integral* because its value depends on the choice of c.

A general **indefinite integral** $I(x)$ is defined as

$$I(x) = \int f(x)\,dx \tag{B.35}$$

where $f(x)$ is called the *integrand* and $f(x) = \dfrac{dI(x)}{dx}$.

For a *general continuous* function $f(x)$, the integral can be described as the area under the curve bounded by $f(x)$ and the x axis, between two specified values of x, say, x_1 and x_2, as in Figure B.14.

The area of the blue element is approximately $f(x_i)\Delta x_i$. If we sum all these area elements from x_1 and x_2 and take the limit of this sum as $\Delta x_i \to 0$, we obtain the *true* area under the curve bounded by $f(x)$ and x, between the limits x_1 and x_2:

$$\text{Area} = \lim_{\Delta x_i \to 0} \sum_i f(x_i)\,\Delta x_i = \int_{x_1}^{x_2} f(x)\,dx \tag{B.36}$$

Integrals of the type defined by Equation B.36 are called **definite integrals.**

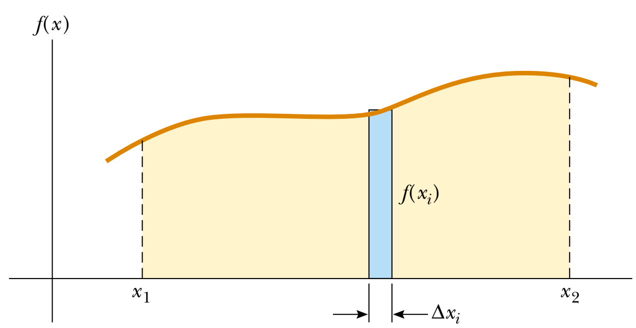


Figure B.14

One common integral that arises in practical situations has the form

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c \qquad (n \neq -1) \tag{B.37}$$

This result is obvious, being that differentiation of the right-hand side with respect to x gives $f(x) = x^n$ directly. If the limits of the integration are known, this integral becomes a *definite integral* and is written

$$\int_{x_1}^{x_2} x^n\,dx = \frac{x_2^{\,n+1} - x_1^{\,n+1}}{n+1} \qquad (n \neq -1) \tag{B.38}$$

Examples

1. $\displaystyle\int_0^a x^2\,dx = \frac{x^3}{3}\Big]_0^a = \frac{a^3}{3}$
2. $\displaystyle\int_0^b x^{3/2}\,dx = \frac{x^{5/2}}{5/2}\Big]_0^b = \frac{2}{5}\,b^{5/2}$
3. $\displaystyle\int_3^5 x\,dx = \frac{x^2}{2}\Big]_3^5 = \frac{5^2-3^2}{2} = 8$

Partial Integration

Sometimes it is useful to apply the method of *partial integration* (also called "integrating by parts") to evaluate certain integrals. The method uses the property that

$$\int u\,dv = uv - \int v\,du \tag{B.39}$$

where u and v are *carefully* chosen so as to reduce a complex integral to a simpler one. In many cases, several reductions have to be made. Consider the function

$$I(x) = \int x^2 e^x\,dx$$

This can be evaluated by integrating by parts twice. First, if we choose $u = x^2$, $v = e^x$, we get

$$\int x^2 e^x\,dx = \int x^2\,d(e^x) = x^2 e^x - 2\int e^x x\,dx + c_1$$

Now, in the second term, choose $u = x$, $v = e^x$, which gives

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2\int e^x\,dx + c_1$$

or

$$\int x^2 e^x\,dx = x^2 e^x - 2xe^x + 2e^x + c_2$$

The Perfect Differential

Another useful method to remember is the use of the *perfect differential,* in which we look for a change of variable such that the differential of the function is the differential of the independent variable appearing in the integrand. For example, consider the integral

$$I(x) = \int \cos^2 x \sin x\,dx$$

This becomes easy to evaluate if we rewrite the differential as $d(\cos x) = -\sin x\,dx$. The integral then becomes

$$\int \cos^2 x \sin x\,dx = -\int \cos^2 x\,d(\cos x)$$

If we now change variables, letting $y = \cos x$, we obtain

$$\int \cos^2 x \sin x\,dx = -\int y^2 dy = -\frac{y^3}{3} + c = -\frac{\cos^3 x}{3} + c$$

Table B.5 lists some useful indefinite integrals. Table B.6 gives Gauss's probability integral and other definite integrals. A more complete list can be found in various handbooks, such as *The Handbook of Chemistry and Physics,* CRC Press.

TABLE B.5 Some Indefinite Integrals (An arbitrary constant should be added to each of these integrals.)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \qquad (\text{provided } n \neq -1)$$

$$\int \frac{dx}{x} = \int x^{-1}\,dx = \ln x$$

$$\int \frac{dx}{a+bx} = \frac{1}{b}\ln(a+bx)$$

$$\int \frac{x\,dx}{a+bx} = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$$

$$\int \frac{dx}{x(x+a)} = -\frac{1}{a}\ln\frac{x+a}{x}$$

$$\int \frac{dx}{(a+bx)^2} = -\frac{1}{b(a+bx)}$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\frac{a+x}{a-x} \qquad (a^2 - x^2 > 0)$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\frac{x-a}{x+a} \qquad (x^2 - a^2 > 0)$$

$$\int \frac{x\,dx}{a^2 \pm x^2} = \pm\tfrac{1}{2}\ln(a^2 \pm x^2)$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} = -\cos^{-1}\frac{x}{a} \qquad (a^2 - x^2 > 0)$$

$$\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2})$$

$$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$$

$$\int \frac{x\,dx}{\sqrt{x^2 \pm a^2}} = \sqrt{x^2 \pm a^2}$$

$$\int \sqrt{a^2-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}\right)$$

$$\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{3}(a^2-x^2)^{3/2}$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \tfrac{1}{2}[x\sqrt{x^2 \pm a^2} \pm a^2\ln(x + \sqrt{x^2 \pm a^2})]$$

$$\int x(\sqrt{x^2 \pm a^2})\,dx = \tfrac{1}{3}(x^2 \pm a^2)^{3/2}$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}$$

$$\int \ln ax\,dx = (x\ln ax) - x$$

$$\int xe^{ax}\,dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int \frac{dx}{a+be^{cx}} = \frac{x}{a} - \frac{1}{ac}\ln(a+be^{cx})$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax$$

$$\int \tan ax\,dx = \frac{1}{a}\ln(\cos ax) = \frac{1}{a}\ln(\sec ax)$$

$$\int \cot ax\,dx = \frac{1}{a}\ln(\sin ax)$$

$$\int \sec ax\,dx = \frac{1}{a}\ln(\sec ax + \tan ax) = \frac{1}{a}\ln\left[\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right]$$

$$\int \csc ax\,dx = \frac{1}{a}\ln(\csc ax - \cot ax) = \frac{1}{a}\ln\left(\tan\frac{ax}{2}\right)$$

$$\int \sin^2 ax\,dx = \frac{x}{2} - \frac{\sin 2ax}{4a}$$

$$\int \cos^2 ax\,dx = \frac{x}{2} + \frac{\sin 2ax}{4a}$$

$$\int \frac{dx}{\sin^2 ax} = -\frac{1}{a}\cot ax$$

$$\int \frac{dx}{\cos^2 ax} = \frac{1}{a}\tan ax$$

$$\int \tan^2 ax\,dx = \frac{1}{a}(\tan ax) - x$$

$$\int \cot^2 ax\,dx = -\frac{1}{a}(\cot ax) - x$$

$$\int \sin^{-1} ax\,dx = x(\sin^{-1} ax) + \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \cos^{-1} ax\,dx = x(\cos^{-1} ax) - \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{\sqrt{x^2+a^2}}$$

TABLE B.6 Gauss's Probability Integral and Other Definite Integrals

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$

$$I_0 = \int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \text{(Gauss's probability integral)}$$

$$I_1 = \int_0^\infty x e^{-ax^2}\,dx = \frac{1}{2a}$$

$$I_2 = \int_0^\infty x^2 e^{-ax^2}\,dx = -\frac{dI_0}{da} = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$I_3 = \int_0^\infty x^3 e^{-ax^2}\,dx = -\frac{dI_1}{da} = \frac{1}{2a^2}$$

$$I_4 = \int_0^\infty x^4 e^{-ax^2}\,dx = \frac{d^2 I_0}{da^2} = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$

$$I_5 = \int_0^\infty x^5 e^{-ax^2}\,dx = \frac{d^2 I_1}{da^2} = \frac{1}{a^3}$$

$$\vdots$$

$$I_{2n} = (-1)^n \frac{d^n}{da^n} I_0$$

$$I_{2n+1} = (-1)^n \frac{d^n}{da^n} I_1$$

APPENDIX C • *Periodic Table of the Elements*

Symbol — **Ca** 20 — Atomic number
Atomic mass † — 40.08
$4s^2$ — Electron configuration

Group I	Group II	Transition elements						
H 1 1.008 0 $1s^1$								
Li 3 6.94 $2s^1$	**Be** 4 9.012 $2s^2$							
Na 11 22.99 $3s^1$	**Mg** 12 24.31 $3s^2$							
K 19 39.102 $4s^1$	**Ca** 20 40.08 $4s^2$	**Sc** 21 44.96 $3d^14s^2$	**Ti** 22 47.90 $3d^24s^2$	**V** 23 50.94 $3d^34s^2$	**Cr** 24 51.996 $3d^54s^1$	**Mn** 25 54.94 $3d^54s^2$	**Fe** 26 55.85 $3d^64s^2$	**Co** 27 58.93 $3d^74s^2$
Rb 37 85.47 $5s^1$	**Sr** 38 87.62 $5s^2$	**Y** 39 88.906 $4d^15s^2$	**Zr** 40 91.22 $4d^25s^2$	**Nb** 41 92.91 $4d^45s^1$	**Mo** 42 95.94 $4d^55s^1$	**Tc** 43 (99) $4d^55s^2$	**Ru** 44 101.1 $4d^75s^1$	**Rh** 45 102.91 $4d^85s^1$
Cs 55 132.91 $6s^1$	**Ba** 56 137.34 $6s^2$	57-71*	**Hf** 72 178.49 $5d^26s^2$	**Ta** 73 180.95 $5d^36s^2$	**W** 74 183.85 $5d^46s^2$	**Re** 75 186.2 $5d^56s^2$	**Os** 76 190.2 $5d^66s^2$	**Ir** 77 192.2 $5d^76s^2$
Fr 87 (223) $7s^1$	**Ra** 88 (226) $7s^2$	89-103**	**Rf** 104 (261) $6d^27s^2$	**Db** 105 (262) $6d^37s^2$	**Sg** 106 (263)	**Bh** 107 (262)	**Hs** 108 (265)	**Mt** 109 (266)

*Lanthanide series	**La** 57 138.91 $5d^16s^2$	**Ce** 58 140.12 $5d^14f^16s^2$	**Pr** 59 140.91 $4f^36s^2$	**Nd** 60 144.24 $4f^46s^2$	**Pm** 61 (147) $4f^56s^2$	**Sm** 62 150.4 $4f^66s^2$
**Actinide series	**Ac** 89 (227) $6d^17s^2$	**Th** 90 (232) $6d^27s^2$	**Pa** 91 (231) $5f^26d^17s^2$	**U** 92 (238) $5f^36d^17s^2$	**Np** 93 (239) $5f^46d^17s^2$	**Pu** 94 (239) $5f^66d^07s^2$

Atomic mass values given are averaged over isotopes in the percentages in which they exist in nature.
† For an unstable element, mass number of the most stable known isotope is given in parentheses.
†† Elements 110, 111, 112, and 114 have not yet been named.
††† For a description of the atomic data, visit **physics.nist.gov/atomic**

			Group III	Group IV	Group V	Group VI	Group VII	Group 0
							H 1 1.008 0 $1s^1$	**He** 2 4.002 6 $1s^2$
			B 5 10.81 $2p^1$	**C** 6 12.011 $2p^2$	**N** 7 14.007 $2p^3$	**O** 8 15.999 $2p^4$	**F** 9 18.998 $2p^5$	**Ne** 10 20.18 $2p^6$
			Al 13 26.98 $3p^1$	**Si** 14 28.09 $3p^2$	**P** 15 30.97 $3p^3$	**S** 16 32.06 $3p^4$	**Cl** 17 35.453 $3p^5$	**Ar** 18 39.948 $3p^6$
Ni 28 58.71 $3d^8 4s^2$	**Cu** 29 63.54 $3d^{10} 4s^1$	**Zn** 30 65.37 $3d^{10} 4s^2$	**Ga** 31 69.72 $4p^1$	**Ge** 32 72.59 $4p^2$	**As** 33 74.92 $4p^3$	**Se** 34 78.96 $4p^4$	**Br** 35 79.91 $4p^5$	**Kr** 36 83.80 $4p^6$
Pd 46 106.4 $4d^{10}$	**Ag** 47 107.87 $4d^{10} 5s^1$	**Cd** 48 112.40 $4d^{10} 5s^2$	**In** 49 114.82 $5p^1$	**Sn** 50 118.69 $5p^2$	**Sb** 51 121.75 $5p^3$	**Te** 52 127.60 $5p^4$	**I** 53 126.90 $5p^5$	**Xe** 54 131.30 $5p^6$
Pt 78 195.09 $5d^9 6s^1$	**Au** 79 196.97 $5d^{10} 6s^1$	**Hg** 80 200.59 $5d^{10} 6s^2$	**Tl** 81 204.37 $6p^1$	**Pb** 82 207.2 $6p^2$	**Bi** 83 208.98 $6p^3$	**Po** 84 (210) $6p^4$	**At** 85 (218) $6p^5$	**Rn** 86 (222) $6p^6$
110†† (269)	111†† (272)	112†† (277)		114†† (289)				

Eu 63 152.0 $4f^7 6s^2$	**Gd** 64 157.25 $5d^1 4f^7 6s^2$	**Tb** 65 158.92 $5d^1 4f^8 6s^2$	**Dy** 66 162.50 $4f^{10} 6s^2$	**Ho** 67 164.93 $4f^{11} 6s^2$	**Er** 68 167.26 $4f^{12} 6s^2$	**Tm** 69 168.93 $4f^{13} 6s^2$	**Yb** 70 173.04 $4f^{14} 6s^2$	**Lu** 71 174.97 $5d^1 4f^{14} 6s^2$
Am 95 (243) $5f^7 6d^0 7s^2$	**Cm** 96 (245) $5f^7 6d^1 7s^2$	**Bk** 97 (247) $5f^8 6d^1 7s^2$	**Cf** 98 (249) $5f^{10} 6d^0 7s^2$	**Es** 99 (254) $5f^{11} 6d^0 7s^2$	**Fm** 100 (253) $5f^{12} 6d^0 7s^2$	**Md** 101 (255) $5f^{13} 6d^0 7s^2$	**No** 102 (255) $6d^0 7s^2$	**Lr** 103 (257) $6d^1 7s^2$

APPENDIX D • SI Units

TABLE D.1 SI Units

	SI Base Unit	
Base Quantity	**Name**	**Symbol**
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mole	mol
Luminous intensity	Candela	cd

TABLE D.2 Some Derived SI Units

Quantity	Name	Symbol	Expression in Terms of Base Units	Expression in Terms of Other SI Units
Plane angle	radian	rad	m/m	
Frequency	hertz	Hz	s^{-1}	
Force	newton	N	$kg \cdot m/s^2$	J/m
Pressure	pascal	Pa	$kg/m \cdot s^2$	N/m^2
Energy; work	joule	J	$kg \cdot m^2/s^2$	$N \cdot m$
Power	watt	W	$kg \cdot m^2/s^3$	J/s
Electric charge	coulomb	C	$A \cdot s$	
Electric potential	volt	V	$kg \cdot m^2/A \cdot s^3$	W/A
Capacitance	farad	F	$A^2 \cdot s^4/kg \cdot m^2$	C/V
Electric resistance	ohm	Ω	$kg \cdot m^2/A^2 \cdot s^3$	V/A
Magnetic flux	weber	Wb	$kg \cdot m^2/A \cdot s^2$	$V \cdot s$
Magnetic field intensity	tesla	T	$kg/A \cdot s^2$	
Inductance	henry	H	$kg \cdot m^2/A^2 \cdot s^2$	$T \cdot m^2/A$

APPENDIX E • *Nobel Prizes*

All Nobel Prizes in physics are listed (and marked with a P), as well as relevant Nobel Prizes in Chemistry (C). The key dates for some of the scientific work are supplied; they often antedate the prize considerably.

1901 (P) *Wilhelm Roentgen* for discovering x-rays (1895).
1902 (P) *Hendrik A. Lorentz* for predicting the Zeeman effect and *Pieter Zeeman* for discovering the Zeeman effect, the splitting of spectral lines in magnetic fields.
1903 (P) *Antoine-Henri Becquerel* for discovering radioactivity (1896) and *Pierre* and *Marie Curie* for studying radioactivity.
1904 (P) *Lord Rayleigh* for studying the density of gases and discovering argon.
(C) *William Ramsay* for discovering the inert gas elements helium, neon, xenon, and krypton, and placing them in the periodic table.
1905 (P) *Philipp Lenard* for studying cathode rays, electrons (1898–1899).
1906 (P) *J. J. Thomson* for studying electrical discharge through gases and discovering the electron (1897).
1907 (P) *Albert A. Michelson* for inventing optical instruments and measuring the speed of light (1880s).
1908 (P) *Gabriel Lippmann* for making the first color photographic plate, using interference methods (1891).
(C) *Ernest Rutherford* for discovering that atoms can be broken apart by alpha rays and for studying radioactivity.
1909 (P) *Guglielmo Marconi* and *Carl Ferdinand Braun* for developing wireless telegraphy.
1910 (P) *Johannes D. van der Waals* for studying the equation of state for gases and liquids (1881).
1911 (P) *Wilhelm Wien* for discovering Wien's law giving the peak of a blackbody spectrum (1893).
(C) *Marie Curie* for discovering radium and polonium (1898) and isolating radium.
1912 (P) *Nils Dalén* for inventing automatic gas regulators for lighthouses.
1913 (P) *Heike Kamerlingh Onnes* for the discovery of superconductivity and liquefying helium (1908).
1914 (P) *Max T. F. von Laue* for studying x-rays from their diffraction by crystals, showing that x-rays are electromagnetic waves (1912).
(C) *Theodore W. Richards* for determining the atomic weights of sixty elements, indicating the existence of isotopes.
1915 (P) *William Henry Bragg* and *William Lawrence Bragg,* his son, for studying the diffraction of x-rays in crystals.
1917 (P) *Charles Barkla* for studying atoms by x-ray scattering (1906).
1918 (P) *Max Planck* for discovering energy quanta (1900).
1919 (P) *Johannes Stark,* for discovering the Stark effect, the splitting of spectral lines in electric fields (1913).

1920 (P) *Charles-Édouard Guillaume* for discovering invar, a nickel-steel alloy with low coefficient of expansion.
(C) *Walther Nernst* for studying heat changes in chemical reactions and formulating the third law of thermodynamics (1918).
1921 (P) *Albert Einstein* for explaining the photoelectric effect and for his services to theoretical physics (1905).
(C) *Frederick Soddy* for studying the chemistry of radioactive substances and discovering isotopes (1912).
1922 (P) *Niels Bohr* for his model of the atom and its radiation (1913).
(C) *Francis W. Aston* for using the mass spectrograph to study atomic weights, thus discovering 212 of the 287 naturally occurring isotopes.
1923 (P) *Robert A. Millikan* for measuring the charge on an electron (1911) and for studying the photoelectric effect experimentally (1914).
1924 (P) *Karl M. G. Siegbahn* for his work in x-ray spectroscopy.
1925 (P) *James Franck* and *Gustav Hertz* for discovering the Franck-Hertz effect in electron-atom collisions.
1926 (P) *Jean-Baptiste Perrin* for studying Brownian motion to validate the discontinuous structure of matter and measure the size of atoms.
1927 (P) *Arthur Holly Compton* for discovering the Compton effect on x-rays, their change in wavelength when they collide with matter (1922), and *Charles T. R. Wilson* for inventing the cloud chamber, used to study charged particles (1906).
1928 (P) *Owen W. Richardson* for studying the thermionic effect and electrons emitted by hot metals (1911).
1929 (P) *Louis Victor de Broglie* for discovering the wave nature of electrons (1923).
1930 (P) *Chandrasekhara Venkata Raman* for studying Raman scattering, the scattering of light by atoms and molecules with a change in wavelength (1928).
1932 (P) *Werner Heisenberg* for creating quantum mechanics (1925).
1933 (P) *Erwin Schrödinger* and *Paul A. M. Dirac* for developing wave mechanics (1925) and relativistic quantum mechanics (1927).
(C) *Harold Urey* for discovering heavy hydrogen, deuterium (1931).
1935 (P) *James Chadwick* for discovering the neutron (1932).
(C) *Irène* and *Frédéric Joliot-Curie* for synthesizing new radioactive elements.
1936 (P) *Carl D. Anderson* for discovering the positron in particular and antimatter in general (1932) and *Victor F. Hess* for discovering cosmic rays.
(C) *Peter J. W. Debye* for studying dipole moments and diffraction of x-rays and electrons in gases.
1937 (P) *Clinton Davisson* and *George Thomson* for discovering the diffraction of electrons by crystals, confirming de Broglie's hypothesis (1927).
1938 (P) *Enrico Fermi* for producing the transuranic radioactive elements by neutron irradiation (1934–1937).
1939 (P) *Ernest O. Lawrence* for inventing the cyclotron.
1943 (P) *Otto Stern* for developing molecular-beam studies (1923), and using them to discover the magnetic moment of the proton (1933).
1944 (P) *Isidor I. Rabi* for discovering nuclear magnetic resonance in atomic and molecular beams.
(C) *Otto Hahn* for discovering nuclear fission (1938).
1945 (P) *Wolfgang Pauli* for discovering the exclusion principle (1924).
1946 (P) *Percy W. Bridgman* for studying physics at high pressures.
1947 (P) *Edward V. Appleton* for studying the ionosphere.

1948 (P) *Patrick M. S. Blackett* for studying nuclear physics with cloud-chamber photographs of cosmic-ray interactions.
1949 (P) *Hideki Yukawa* for predicting the existence of mesons (1935).
1950 (P) *Cecil F. Powell* for developing the method of studying cosmic rays with photographic emulsions and discovering new mesons.
1951 (P) *John D. Cockcroft* and *Ernest T. S. Walton* for transmuting nuclei in an accelerator (1932).
(C) *Edwin M. McMillan* for producing neptunium (1940) and *Glenn T. Seaborg* for producing plutonium (1941) and further transuranic elements.
1952 (P) *Felix Bloch* and *Edward Mills Purcell* for discovering nuclear magnetic resonance in liquids and gases (1946).
1953 (P) *Frits Zernike* for inventing the phase-contrast microscope, which uses interference to provide high contrast.
1954 (P) *Max Born* for interpreting the wave function as a probability (1926) and other quantum-mechanical discoveries and *Walther Bothe* for developing the coincidence method to study subatomic particles (1930–1931), producing, in particular, the particle interpreted by Chadwick as the neutron.
1955 (P) *Willis E. Lamb, Jr.,* for discovering the Lamb shift in the hydrogen spectrum (1947) and *Polykarp Kusch* for determining the magnetic moment of the electron (1947).
1956 (P) *John Bardeen, Walter H. Brattain,* and *William Shockley* for inventing the transistor (1956).
1957 (P) *T.-D. Lee* and *C.-N. Yang* for predicting that parity is not conserved in beta decay (1956).
1958 (P) *Pavel A. Čerenkov* for discovering Čerenkov radiation (1935) and *Ilya M. Frank* and *Igor Tamm* for interpreting it (1937).
1959 (P) *Emilio G. Segrè* and *Owen Chamberlain* for discovering the antiproton (1955).
1960 (P) *Donald A. Glaser* for inventing the bubble chamber to study elementary particles (1952).
(C) *Willard Libby* for developing radiocarbon dating (1947).
1961 (P) *Robert Hofstadter* for discovering internal structure in protons and neutrons and *Rudolf L. Mössbauer* for discovering the Mössbauer effect of recoilless gamma-ray emission (1957).
1962 (P) *Lev Davidovich Landau* for studying liquid helium and other condensed matter theoretically.
1963 (P) *Eugene P. Wigner* for applying symmetry principles to elementary-particle theory and *Maria Goeppert Mayer* and *J. Hans D. Jensen* for studying the shell model of nuclei (1947).
1964 (P) *Charles H. Townes, Nikolai G. Basov,* and *Alexandr M. Prokhorov* for developing masers (1951–1952) and lasers.
1965 (P) *Sin-itiro Tomonaga, Julian S. Schwinger,* and *Richard P. Feynman* for developing quantum electrodynamics (1948).
1966 (P) *Alfred Kastler* for his optical methods of studying atomic energy levels.
1967 (P) *Hans Albrecht Bethe* for discovering the routes of energy production in stars (1939).
1968 (P) *Luis W. Alvarez* for discovering resonance states of elementary particles.
1969 (P) *Murray Gell-Mann* for classifying elementary particles (1963).
1970 (P) *Hannes Alfvén* for developing magnetohydrodynamic theory and *Louis Eugène Félix Néel* for discovering antiferromagnetism and ferrimagnetism (1930s).

1971 (P) *Dennis Gabor* for developing holography (1947).
(C) *Gerhard Herzberg* for studying the structure of molecules spectroscopically.

1972 (P) *John Bardeen, Leon N. Cooper,* and *John Robert Schrieffer* for explaining superconductivity (1957).

1973 (P) *Leo Esaki* for discovering tunneling in semiconductors, *Ivar Giaever* for discovering tunneling in superconductors, and *Brian D. Josephson* for predicting the Josephson effect, which involves tunneling of paired electrons (1958–1962).

1974 (P) *Anthony Hewish* for discovering pulsars and *Martin Ryle* for developing radio interferometry.

1975 (P) *Aage N. Bohr, Ben R. Mottelson,* and *James Rainwater* for discovering why some nuclei take asymmetric shapes.

1976 (P) *Burton Richter* and *Samuel C. C. Ting* for discovering the J/psi particle, the first charmed particle (1974).

1977 (P) *John H. Van Vleck, Nevill F. Mott,* and *Philip W. Anderson* for studying solids quantum-mechanically.
(C) *Ilya Prigogine* for extending thermodynamics to show how life could arise in the face of the second law.

1978 (P) *Arno A. Penzias* and *Robert W. Wilson* for discovering the cosmic background radiation (1965) and *Pyotr Kapitsa* for his studies of liquid helium.

1979 (P) *Sheldon L. Glashow, Abdus Salam,* and *Steven Weinberg* for developing the theory that unified the weak and electromagnetic forces (1958–1971).

1980 (P) *Val Fitch* and *James W. Cronin* for discovering CP (charge-parity) violation (1964), which possibly explains the cosmological dominance of matter over antimatter.

1981 (P) *Nicolaas Bloembergen* and *Arthur L. Schawlow* for developing laser spectroscopy and *Kai M. Siegbahn* for developing high-resolution electron spectroscopy (1958).

1982 (P) *Kenneth G. Wilson* for developing a method of constructing theories of phase transitions to analyze critical phenomena.

1983 (P) *William A. Fowler* for theoretical studies of astrophysical nucleosynthesis and *Subramanyan Chandrasekhar* for studying physical processes of importance to stellar structure and evolution, including the prediction of white dwarf stars (1930).

1984 (P) *Carlo Rubbia* for discovering the W and Z particles, verifying the electroweak unification, and *Simon van der Meer,* for developing the method of stochastic cooling of the CERN beam that allowed the discovery (1982–1983).

1985 (P) *Klaus von Klitzing* for the quantized Hall effect, relating to conductivity in the presence of a magnetic field (1980).

1986 (P) *Ernst Ruska* for inventing the electron microscope (1931), and *Gerd Binnig* and *Heinrich Rohrer* for inventing the scanning-tunneling electron microscope (1981).

1987 (P) *J. Georg Bednorz* and *Karl Alex Müller* for the discovery of high temperature superconductivity (1986).

1988 (P) *Leon M. Lederman, Melvin Schwartz,* and *Jack Steinberger* for a collaborative experiment that led to the development of a new tool for studying the weak nuclear force, which affects the radioactive decay of atoms.

1989 (P) *Norman Ramsay* (U.S.) for various techniques in atomic physics; and *Hans Dehmelt* (U.S.) and *Wolfgang Paul* (Germany) for the development of techniques for trapping single charge particles.

1990 (P) *Jerome Friedman, Henry Kendall* (both U.S.), and *Richard Taylor* (Canada) for experiments important to the development of the quark model.

1991 (P) *Pierre-Gilles de Gennes* for discovering that methods developed for studying order phenomena in simple systems can be generalized to more complex forms of matter, in particular to liquid crystals and polymers.

1992 (P) *George Charpak* for developing detectors that trace the paths of evanescent subatomic particles produced in particle accelerators.

1993 (P) *Russell Hulse* and *Joseph Taylor* for discovering evidence of gravitational waves.

1994 (P) *Bertram N. Brockhouse* and *Clifford G. Shull* for pioneering work in neutron scattering.

1995 (P) *Martin L. Perl* and *Frederick Reines* for discovering the tau particle and the neutrino, respectively.

1996 (P) *David M. Lee, Douglas C. Osheroff,* and *Robert C. Richardson* for developing a superfluid using helium-3.

1997 (P) *Steven Chu, Claude Cohen-Tannoudji,* and *William D. Phillips* for developing methods to cool and trap atoms with laser light.

1998 (P) *Robert B. Laughlin, Horst L. Störmer,* and *Daniel C. Tsui* for discovering a new form of quantum fluid with fractionally charged excitations.